The Poisson Geometry of Replicator Dynamics

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Table of contents

Poisson geometry

Why?

- Dynamical system describing evolution of distribution of frequencies
- Discrete probability distribution

$$x \in \Delta^n \subset \mathbb{R}^{n+1} = \{x \in \mathbb{R}^{n+1} : \sum_i x^i = 1, x^i \geq 0\} \tag{1}$$

- No symplectic structure on odd faces

Poisson algebra

$(V, \circ, \{\cdot,\cdot\})$ vector space with two bilinear operations

- $(V, K, \circ)$ associative algebra
- $(V, K, \{\cdot,\cdot\})$ Lie algebra (a.s. and Jacobi)
- $\{\cdot,\cdot\}$ derivation with respect to $\circ$ in both arguments, namely for any fixed $u \in V$ the map $\{u,\cdot\} : V \to V$ fulfills

$$\{u, a \circ b\} = \{u, a\} \circ b + a \circ \{u, b\} \tag{2}$$

for any $a, b \in V$, and similarly for $\{\cdot, u\}$.

The map $\{\cdot,\cdot\}$ is called Poisson bracket.

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Poisson manifold

- Smooth manifold M with a Poisson bracket $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ making $(C^\infty(M), \{\cdot,\cdot\})$ a Poisson algebra.
- Poisson bivector: π antisymmetric $(2,0)$ tensor field[1]

$$\{f,g\} = \pi(\mathrm{d}f, \mathrm{d}g) \tag{3}$$

$$\sum_{\text{cyclic } i,j,h} \pi^{ik}\,\partial_k \pi^{jh} = 0 \qquad \text{(Jacobi)}$$

[1] [Vai94, p. 4][DZ05, p. 6][LM87, p. 109]

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Remarks

- $[\pi, \pi]_S = 0$ *Schouten-Nijenhuis* bracket[2]
- Symplectic manifold is Poisson[3]

$$\begin{aligned} \{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) &\to C^\infty(M) \\ (f,g) &\longmapsto \{f,g\} = \omega(X_f, X_g) = \pi(\mathrm{d}f, \mathrm{d}g) \end{aligned} \tag{4}$$

[2] [Vai94, p. 6] [DZ05, p. 27][BV88]

[3] Sign convention $\iota_{X_f}\omega = -\mathrm{d}f$

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Poisson Hamiltonian vector field

- Sharp homomorphism $\sharp : \Omega(M) \to \tau(M)$ defined also if π degenerate
- Hamiltonian vector field

$$X_f = (\mathrm{d}f)^\sharp = \pi(\mathrm{d}f, \cdot) \tag{5}$$

$$X_f f = \pi(\mathrm{d}f, \mathrm{d}f) \equiv 0 \tag{6}$$

Nondegenerate Poisson manifold is symplectic

$$\omega(X, Y) = \pi(X^{\flat_\pi}, Y^{\flat_\pi}), \quad \forall X, Y \in \tau(M) \tag{7}$$

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- bivectors are F-related

$$\pi_M(f \circ F, g \circ F) = \pi_N(f, g) \circ F, \quad \forall f, g \in C^\infty(N) \tag{8}$$

- pullback is Lie algebra homomorphism

$$\{F^* f, F^* g\}_M = F^* \{f, g\}_N \tag{9}$$

Poisson vector field $\Leftarrow$ Hamiltonian vector field[4]

$$\mathscr{L}_X \pi = 0 \tag{10}$$

Local flow $\Theta_t(p)$ is Poisson diffeomorphism

[4] LM87, p. 122.

Poisson morphism $F : (M, \pi_M) \to (N, \pi_N)$

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Example

$$F : \mathbb{R}^4 \to \mathbb{R}^2$$

$$(q^1, p^1, q^2, p^2) \longmapsto (x, y) = (q^1, p^1)$$

$$\pi_4^{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \pi_2^{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$J \pi_M J^T = \pi_N \circ F$$

How does the motion along Hamiltonian vector fields look like on a Poisson manifold?

- Poisson submanifold $\iota : S \to M$ such that every Hamiltonian vector field is tangent to S
- S Poisson manifold such that ι is Poisson morphism[5]
- Characteristic space

$$C_p = \mathrm{Im}\left(\sharp_p\right) = \sharp_p\left(T_p^*M\right) \subset T_pM \tag{11}$$

[5] Mei17, p. 19.

Symplectic foliation of a Poisson manifold

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Theorem (Symplectic foliation)

The characteristic distribution of a Poisson manifold is a smooth generalized distribution spanned by Hamiltonian vector fields. It is integrable, and its leaves S are nondegenerate Poisson submanifolds[6].

[6][OR04, p. 131][LM87, p. 130]

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Symplectic foliation of a Poisson manifold

- Every Poisson manifold is a union of disjoint immersed symplectic submanifolds, the immersion being a Poisson morphism.
- Two points belong to the same leaf if and only if they can be connected by a piecewise-smooth curve consisting of integral curves of Hamiltonian vector fields.
- The dimension of the leaf through a point is the rank of π at that point.

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Poisson Reduction

Problem: Let $\psi : G \times M \to M$ be a smooth action of a Lie group G on a manifold M.

- When is the quotient space a manifold?
- Does the quotient preserve structures existing on M?

Lie groups actions

Recall: the action of G on M is

- *proper*[7] if [some technical condition about compactness], always given in the following
- *free* if all isotropy subgroups are trivial

Furthermore if (M, π) is a Poisson manifold the action of G is

- *Poisson* if the map $g : p \mapsto g \cdot p$ is a Poisson morphism for all $g \in G, p \in M$

[7] Lee12, p. 543.

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Theorem (Quotient Manifold)
If a Lie group G acts smoothly, freely and properly on a smooth manifold M then the orbit space ${}^{M}/_{G}$ is a smooth manifold of dimension $\dim M - \dim G$ with unique smooth structure such that the canonical projection is a smooth submersion[8].

What happens removing freeness?

[8] Lee12, p. 544.

Stratified space

Let X be a topological space, and $\mathcal{S} = \{S_i\}_{i \in I}$ a locally finite partition of X such that

- the pieces of $\mathcal{S}$ are locally closed smooth manifolds $S_i \subset X$, called *strata*;
- the strata fulfill a *frontier condition*.[9]

The pair $(X, \mathcal{S})$ is called stratified space, or *stratification* of X.

[9] if a stratum meets the closure of another, the first stratum is contained in the closure of second. $S_i \cap \bar{S}_j \neq \emptyset \Rightarrow S_i \subset \bar{S}_j$. See [OR04, p. 31].

Stratified space - remarks

- collection of manifolds *fitting together nicely*
- in general of different dimensions
- in general not a manifold itself
- e.g. intuitively, a simplex: manifolds = faces

- A SS can be endowed[10] with an appropriate smooth structure and an algebra of smooth functions $C^\infty(X)$.

[10] OR04, p. 32.

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Theorem (Stratification)

If a Lie group G acts smoothly and properly on a smooth manifold M then the orbit space ${}^{M}/_{G}$ is a smooth stratified space[11].

[11] See [OR04, pp. 75,84] for the description of the strata as connected components of the reduced orbit type submanifolds.

Problem: Act with a Lie group G on a manifold M. When is the quotient space a manifold? Does the quotient space preserve structures existing on M?

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Poisson stratified spaces

A Poisson stratification[12] of a topological space X is a smooth stratification $(X, \mathcal{S})$ with a Poisson algebra $(C^\infty(X), \{\cdot, \cdot\})$ such that

- each stratum is a Poisson manifold, and
- each inclusion is a Poisson morphism.

[12] ORF09, p. 1271.

Regular and Singular Poisson Reduction [13]

Theorem (Poisson reduction)

G Lie group acting smoothly and properly on Poisson manifold (M, π).

- *Poisson action: the quotient space is a Poisson stratified space;*
- *Poisson free action: the quotient space is a Poisson manifold;*
- *unique structure such that the canonical projection is Poisson morphism.*

[13] [OR04, p. 364] [ORF09, p. 1273]

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Simplex stratified Poisson structure

The standard simplex

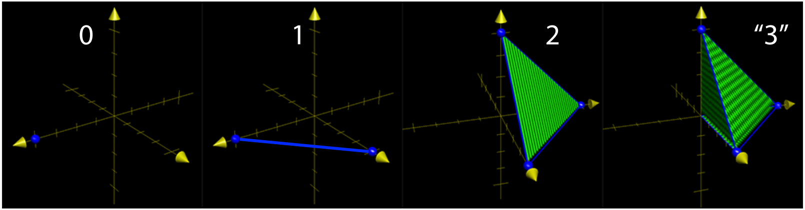

Figure 1: Simplices fully representable in three dimensions

$$\Delta^n \subset \mathbb{R}^{n+1} = \{x \in \mathbb{R}^{n+1} : \sum_i x^i = 1, x^i \geq 0\} \tag{12}$$

Support and faces[14]

$$\Delta^n \subset \mathbb{R}^{n+1} = \{x \in \mathbb{R}^{n+1} : \sum_i x^i = 1, x^i \geq 0\}$$

- $I = \{0, \ldots, n\}$
- $\text{supp}(x) = \{i \in I : x^i > 0\}$
- $J \subset I$ with $d + 1$ elements, $d = 0, \ldots, n$ defines
    - d-face $\mathring{\Delta}^J = \{x \in \Delta^n : \text{supp}(x) = J\}$
    - closed d-face $\Delta^J = \{x \in \Delta^n : \text{supp}(x) \subset J\}$

[14] AL84, p. 235.

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Goal: endow the standard simplex with a stratified Poisson structure via a double reduction procedure.

Regular Poisson reduction

- Quadratic Poisson structure[15] $\{z_i, z_j\} = A_{ij}\, z_i z_j$ on $M = \mathbb{C}^{n+1} - \{0\}$
- A antisymmetric $(n+1) \times (n+1)$ (will be fitness matrix in zero-sum games)

Action of $G = \mathbb{C} - \{0\}$ on M by complex multiplication element-wise

$$\psi_\lambda(z) = \rho e^{i\alpha} \cdot (r_0, \dots r_n, \theta_0, \dots, \theta_n) = (\rho\, r_0, \dots \rho\, r_n, \alpha + \theta_0, \dots, \alpha + \theta_n)$$

Free, proper and Poisson: ${}^{M}/_{G}$ Poisson manifold (*complex projective space*)

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Free, proper and Poisson: ${}^{M}/_{G}$ Poisson manifold (*complex projective space*)

[15] ORF09.

Singular Poisson reduction

Further action of $\mathbb{T}^n$ on ${}^M/_G$:

$$\begin{aligned}\psi_T([z]) = T \cdot [z] &= (e^{i\phi_1}, \ldots, e^{i\phi_n}) \cdot [(z_0, z_1, \ldots, z_n)] \\ &= [(z_0, e^{i\phi_1} z_1, \ldots, e^{i\phi_n} z_n)]\end{aligned} \tag{13}$$

- well defined for any representative element of the class
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$$[(r_i, \theta_i)] \longmapsto [(r_i)]$$

$$\xi : \mathbb{C}P(n) \to \Delta^n \subset \mathbb{R}^{n+1}$$

$$[z] \longmapsto \left(\frac{r_0^2}{r_0^2 + \cdots + r_n^2}, \cdots, \frac{r_n^2}{r_0^2 + \cdots + r_n^2} \right)$$

- well defined
- onto the standard simplex
- $[z] \sim_\xi [w] \iff [z] \sim_{\mathbb{T}^n} [w]$

$$\Delta^n \cong {}^{\mathbb{C}P(n)}/_{\mathbb{T}^n} \tag{14}$$

The standard simplex is a Poisson stratified space with unique Poisson structure such that the canonical projection is a Poisson morphism.

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The stratified Poisson structure of a simplex

- The strata are precisely the faces of the simplex[16]
- The resulting Poisson structure on Δ^n is

$$\{x^i, x^j\} = x^i x^j \left(A_{ij} - \sum_h (A_{ih} + A_{hj}) x^h \right) \tag{15}$$

- This actually is a Poisson structure for the whole R^{n+1} such that the faces are Poisson submanifolds.

[16] Isotropy type submanifolds analysis.

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Next step: A zero-sum replicator dynamical system on Δ^n is Hamiltonian with respect to this Poisson structure if it admits an interior fixpoint[17].

[17] [AD14], [AL84]

References for this section

- Encyclopedia on Hamiltonian reduction [OR04]
- Simplex Stratified Poisson structure [ORF09], [AD14]

Evolutionary games and replicator dynamics

- Consider a population composed of interacting individuals;
- each individual has at its disposal a finite set of behaviors, traits, pure strategies to adopt;
- on this choice and via the interaction with other individuals depends his fitness, his well-being, his payoff, measured in some units;
- via some mechanism (inheritance, learning, imitation, mutation, ...) successful strategies spread;
- *how does the average population strategy evolve?*

Normal form games[18]

An N-normal form game (Δ^N, g) is the collection of

- a set of $N+1$ *pure strategies* $\{R_0, \dots, R_N\}$;
- a *game space* $\Delta^N \in \mathbb{R}^{N+1}$
- a *population* of interacting individuals;
- a *payoff function*

$$\begin{aligned} g : \Delta^N \times \Delta^N &\to \mathbb{R} \\ p, q &\longmapsto g(p, q) \end{aligned} \tag{16}$$

A point in game space is called a strategy, and $g(p, q)$ is the payoff *of the strategy p against the strategy q.*

[18] HS98, p. 57.

Strategies and pure strategies

- Pure strategies: belong to some abstract strategy space. Behavior, physical trait, belief, …
- Strategy
  - discrete probability distribution of pure strategies usage for a single individual;
  - distribution of pure strategies in the population.
  - $p^i \geq 0, \sum_i p^i = 1 \Rightarrow p \in \Delta^N$
- Identify abstract *pure* strategy R_i with vertex strategy e_i of simplex
- $p = p^i e_i$

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Interaction

- **Local**: the payoff of an individual employing a certain strategy depends on the outcome of a pairwise encounter with another individual
    - *bilinear* payoff
    - e.g. Hawks and Doves
- **Global**: no actual pairwise encounter occurs; the payoff of a strategy depends on the actual state of the population as a whole
    - *nonlinear* payoff
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Payoff function

$$\begin{aligned} g : \Delta^N \times \Delta^N &\to \mathbb{R} \\ p, q &\longmapsto g(p, q) \end{aligned} \tag{17}$$

- $g(p, q)$ = payoff to use strategy p vs strategy q
- Always linear in first argument: $g(p, q) = g(p^i e_i, q)$ = $\sum_i$ (prob. I use i-th pure strategy)· (payoff of i-th pure strategy vs q)

$$= p^i g(e_i, q) =: p^i g_i(q) \tag{18}$$

- Second?

1 vs 1, many times: random pairwise encounters in population

- q = *your* prob. distribution of pure strategies usage
- linearity in second argument

$g_i(q)$ = payoff of i-th pure strategy vs q =

$\sum_j$ (payoff of i-th pure strategy vs j-th pure strategy) · (prob. you use j-th pure strategy)

$$= g_i(e_j)\, q^j =: g_{ij}\, q^j \tag{19}$$

Payoff matrix $g_{ij} = g(e_i, e_j)$

Hawks and Doves

- Non lethal fights between animals of the same species
- Darwinian fitness i.e. reproductive success
- carefully decide whether to escalate a fight or not

Consider two pure strategies:

- *Dove*: show off and provoke the opponent, but quit if the opponent actually escalates
- *Hawk*: fights until your or your opponent's defeat, no matter what.

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Hawks and Doves

- Avoided fight has no consequences;
- won fight increases fitness by gain G;
- lost fight decreases fitness by cost $C > G$.

	meeting a dove	meeting a hawk
a dove gets	$G/2$	0
a hawk gets	G	$\frac{G-C}{2}$

to be continued...

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to be continued...

No pairwise encounter occurs

- $g(p,q)$ = payoff of a p-strategist in a population with average q-strategy
- Needs *not* be linear in second argument, e.g. *sex ratio*[19]

$$g(p,q) = \frac{p^0}{q^0} + \frac{p^1}{q^1} \tag{20}$$

The more females there are in a population, the less convenient it is to have female offspring.

[19] HS98, pp. 60,65.

Linear payoff in the following; similar results hold, taking care of adding a notion of locality to some definitions[20].

[20] HS98, pp. 63,65.

Set of best replies

- Set of best replies to $q \in \Delta^N$

$$\beta(q) = \{p \in \Delta^N : g(p,q) = \max_{p' \in \Delta^N} g(p', q)\} \tag{21}$$

- Replace $p^0 = 1 - \sum_{i=1}^{n} p^i$

$$g(p,q) = g_0(q) + \sum_{i=1}^{n} p^i \left(g_i(q) - g_0(q)\right) \tag{22}$$

- Non-linear in q does not matter: linear in p...

... so that $\beta(q)$ is a nonempty union faces, containing a vertex at least and the whole simplex at most. A fixed q is the ”inclination” of the payoff function.

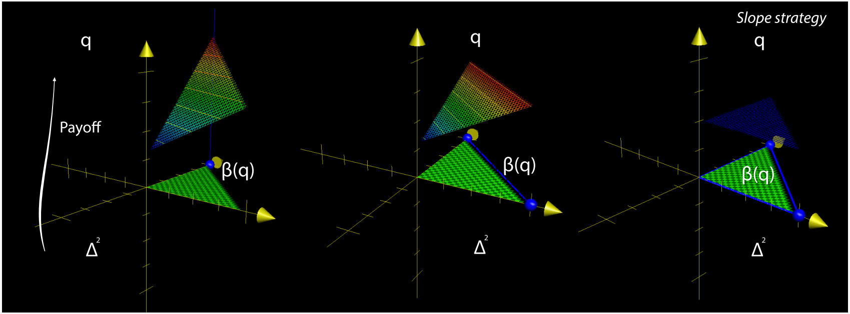

Figure 2: $g(p,q)$ for q fixed, as a function of p for Δ^2

Slope strategy

$$g(p,q) = g_0(q) + \sum_{i=1}^{n} p^i \underbrace{(g_i(q) - g_0(q))}_{\equiv 0 \forall i \text{ at } \hat{q}} \tag{23}$$

- Strategy $\hat{q}$ such that $g(p,\hat{q}) = g_i(\hat{q})$ for all i, no matter which p
- May or may not exists!
- $\beta(\hat{q}) = \Delta^N$
- Slope support:

$$g_i(p) = g_j(p) \quad \forall i,j \in \operatorname{supp}(p)$$

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Static notions of equilibrium

Nash strategy: best reply to itself, i.e. $p \in \beta(p)$

- Strict: $\beta(p) = p$, can only be vertex
- If slope exists, it is Nash $\beta(\hat{q}) = \Delta^N \ni \hat{q}$
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Stability of Nash strategy

- Crucial step from GT to EGT
- If alternative best reply exists, why should one stick with p Nash?

$$g(p,p) = g(q,p), \forall q \in \beta(p) \tag{24}$$

- Stable p Nash: $g(p,q) > g(q,q), \forall q \in \beta(p), q \neq p$
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- Strictly population concept, doesn't make sense 1 vs 1. Indeed..

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Static notions of equilibrium

Stable Nash if and only if[21]

- Evolutionarily stable strategy $\hat{p}$: *if everybody is using it, a mutant minority can not invade*

$$g(\hat{p}, \epsilon p + (1-\epsilon)\hat{p}) > g(p, \epsilon p + (1-\epsilon)\hat{p}) \tag{25}$$

  for all $p \neq \hat{p}$, and for all $0 < \epsilon <$ some positive invasion threshold.

[21] HS98, p. 63.

Back to Hawks and Doves

$p_s = \left(1 - \frac{G}{C}, \frac{G}{C}\right)$ interior slope Nash strategy, stable.

Sweet spot of optimal frequency of engaged fights (in this case precisely the ratio G/C between the gain of a won fight and the cost of a lost one).

- Does a population reach an ES strategy?
- Model the evolution of the average population strategy driven by the interaction between the individuals of the population.
- Dynamical system on the simplex.

(Change notation: $p \to x; g \to f$)

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Replicator dynamics

Basic model for evolution of types frequencies[22]

- type i growth rate = its fitness - average population fitness

$$\dot{x}^i = x^i \left(f_i(x) - \bar{f}(x) \right) \tag{26}$$

$$\bar{f}(x) = \sum_i x^i f_i(x) \tag{27}$$

The replicator vector field is tangent to every face of the simplex $\Delta^N \rightarrow$ no mutations

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Static and dynamic equilibrium

- replicator fixpoint $\iff$ has slope support
- x Evolutionarily stable $\Rightarrow$ x asymptotically stable fixpoint[23]

A lot more to say on the relation between the static and dynamic notions of equilibria, both in the continuous and *discrete* replicator[24], but focus now on antisymmetric fitness function

[23] HS98, p. 70.
[24] Sel91, p. 29.

Zero-sum replicator dynamics

Zero-sum games

- Gain of a player is exactly loss of another
- Extensively studied in classical GT[25]
- Very restrictive assumption for real life applications
- *Discrete* zero-sum replicator: model for gene conversion[26]
- Interesting in its own right for Hamiltonian character
- Related to Rock Paper Scissor games

[25] Sig11, p. 4.
[26] Nag83b; Nag83a.

Three strategies cyclically beating each other (not necessarily zero-sum)

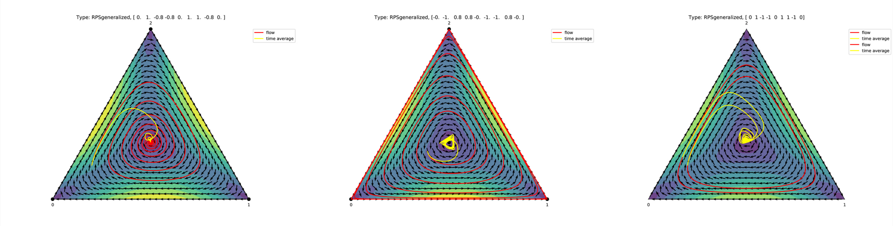

Figure 3: Rock Paper Scissor

Zero sum replicator

- $f_i(x) = \sum_j A_{ij} x^j$
- A antisymmetric fitness matrix
- $\bar{f}(x) \equiv 0$

$$\dot{x}^i = x^i f_i(x) \tag{28}$$

- For ZSG only vertices can be ESS, not very interesting.
- Still two mutually exclusive classes of fixpoints on which the dynamics depends exist[27].

[27] AL84.

Interior and Boundary semi-defined fixpoints

- $E_0 = \{x \in \mathring{\Delta} : f_i(x) = 0\,\forall i\} \equiv$ interior fixpoints
- $E_- = \{x \in \Delta : f_i(x) \leq 0$ with at least one inequality strict$\}$
- $E_+ = \{x \in \Delta : f_i(x) \geq 0$ with at least one inequality strict$\}$

Theorem

These three sets are convex subsets consisting entirely of equilibria. E_+ and E_- are subsets of the boundary of Δ. Precisely one of the following two scenarios occurs[28]

- $E_0 \neq \emptyset, E_+ = \emptyset = E_-$, *interior case;*
- $E_0 = \emptyset, E_+ \neq \emptyset, E_- \neq \emptyset$, *boundary case.*

[28] AL84.

Interior and Boundary semi-defined fixpoints - remark

Upon $A \to -A$

- E_0 is invariant
- $E_\pm$ are exchanged

Hamiltonian dynamics of zero-sum replicator games

- Zero-sum replicator in interior case, $\hat{x} \in E_0$
- The replicator vector field is Hamiltonian with respect to (minus) the simplex Poisson structure[29]
- A Hamiltonian function is $H_{\hat{x}}(x) = -\sum_i \hat{x}^i \ln x^i$
- Convex and coercive with unique strict minimum at $\hat{x}$
- Proof: direct computation $\mathrm{d}H^\sharp = X_{\text{rep}}$ using $f_i(\hat{x}) = 0 \forall i$.

[29] AD14.

Hamiltonian dynamics of zero-sum replicator games

- Closure of interior trajectories is compact invariant set contained in $\mathring{\Delta} - E_0$[30]
- Interior filled with invariant manifolds
- All interior fixpoints are neutrally stable
- coexistence, no strategy goes extinct

[30] AL84, p. 239.

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Hamiltonian dynamics of zero-sum replicator games

- Constant Poisson structure coordinates $y^i = \ln(x^i/x^0)$
- Hamiltonian in new coordinates still convex

$$H(y) = \ln\left(1 + \sum_i e^{y^i}\right) - \sum_i \hat{x}^i y^i \tag{29}$$

- From here: *Convexity methods in Hamiltonian mechanics*; "The dynamics on three-dimensional strictly convex energy surfaces"

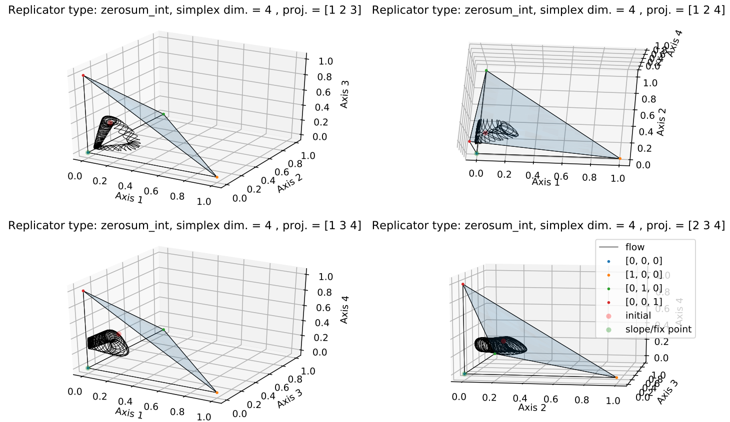

Figure 4: Zero-sum replicator - Hamiltonian interior case

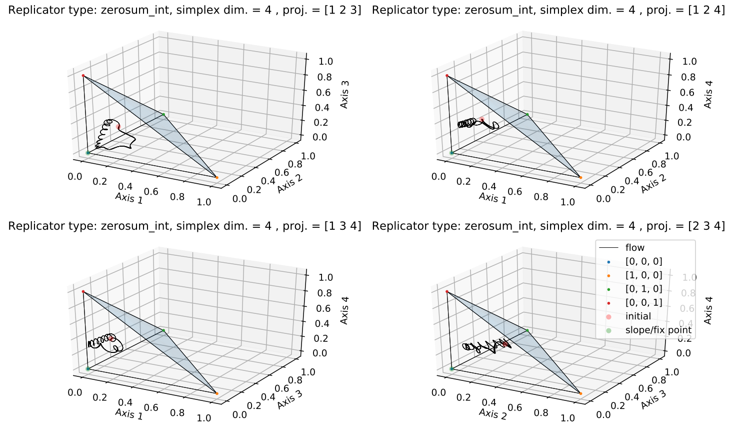

Figure 5: Zero-sum replicator - Hamiltonian interior case

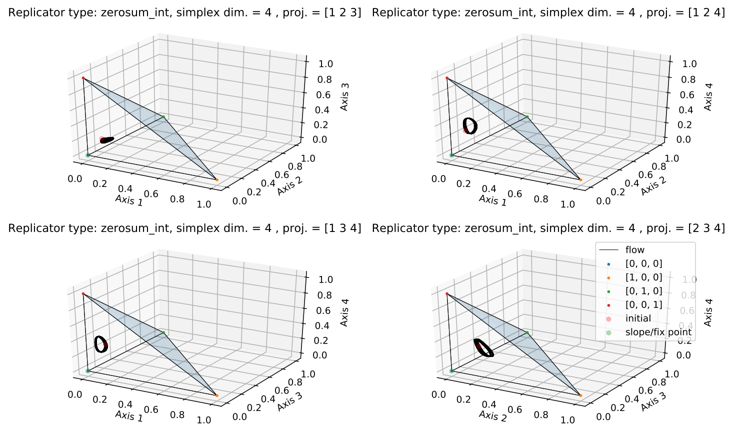

Figure 6: Zero-sum replicator - Hamiltonian interior case

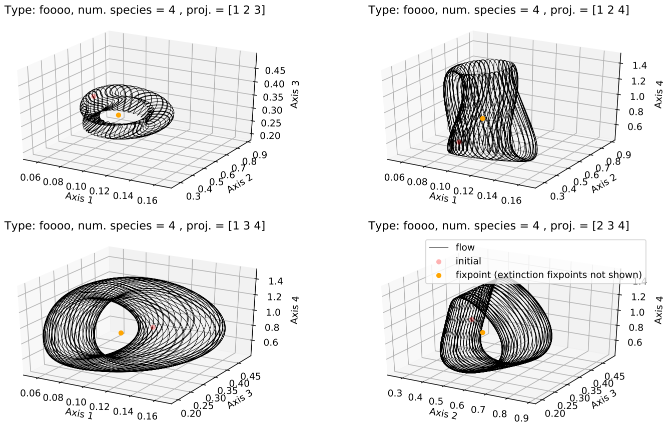

Figure 7: Equivalent LV [HS98, p. 77] - Hamiltonian interior case

Zero sum replicator boundary dynamics

- Zero-sum replicator in boundary[31] case $E_{\pm} \neq \emptyset$
- $e_{-} \in E_{-} \Rightarrow H_{e_{-}}$ *strictly decreasing* along interior trajectories
- The ω-limit of all interior trajectories is a subset of the boundary, in particular

$$J_{-} = \{i \in I : f_i(e_{-}) = 0 \, \forall e_{-} \in E_{-}\}$$

- strategies doing as well as possible against E_{-}

[31] AL84, p. 239.

- Indeed, J_- precisely surviving strategies!

$$\omega(p) \subset \Delta^{J_-} \; \forall p \in \mathring{\Delta} \tag{30}$$

- Points in the closed face

$$\Delta^{J_-} : i \notin J_- \Rightarrow x^i = 0$$

$$\lim_{t\to\infty} x^i(t) = 0 \text{ for all } i \notin J_- \tag{31}$$

Analogue results for E_+ and α-limit, so $A \to -A$ effective time reversal

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Asymptotic Hamiltonian behavior

$$\left(\mathrm{d}H^\sharp\right)^i = X^i_{\mathrm{rep}} - x^i f_i(\hat{x}) + x^i \sum_{h\notin \mathrm{supp}(\hat{x})} x^h f_h(\hat{x})$$

- Extra terms vanish identically if $\hat{x}$ interior equilibrium
- Extra terms vanish asymptotically if $\hat{x} \in E_-$
  - either $f_i(\hat{x}) = 0$ or $x^i \to 0$

- Correspondingly $\mathscr{L}_{X_{\mathrm{rep}}}\pi = \mathscr{L}_{\mathrm{d}H^\sharp}\pi + \mathscr{L}_{X_{\mathrm{bd}}}\pi$
- The first term vanishes identically and the second asymptotically
- It does not look like the second term can be written conformally as $F(x)\pi$ with F vanishing on the future face; still work in progress.

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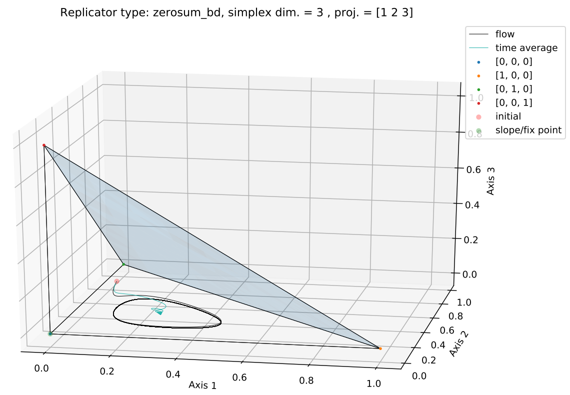

Figure 8: Zero sum replicator, boundary scenario, α-limit

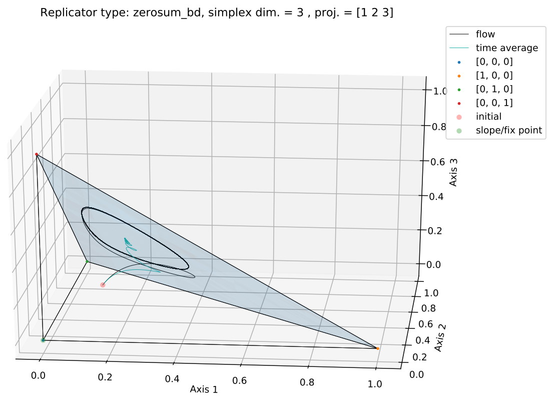

Figure 9: Zero sum replicator, boundary scenario, ω-limit

Zero sum replicator boundary dynamics

$$\begin{bmatrix} 0 & -1.5 & 1.3 & -2.5 \\ 1.5 & 0 & -2.0 & 2.0 \\ -1.3 & 2.0 & 0 & -1.0 \\ 2.5 & -2.0 & 1.0 & 0 \end{bmatrix}$$

- $e_- = (0.2, 0.4, 0.4, 0), \quad f_i(e_-) = (-0.78, 0, 0, 0)$
- $J_- = \{1, 2, 3\}$ surviving in classical RPS dynamics
- 0 extincted

Zero sum replicator boundary dynamics

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- $e_+ = (0.27, 0.31, 0, 0.42), \quad f_i(e_+) = (0, 0, 0, +0.8)$
- 3 invaded in the past

Conclusions

Recap

- A simplex is endowed with a stratified Poisson structure via a reduction procedure; every face is a Poisson manifold.
- The replicator vector field modeling the evolution of the average population strategy is tangent to every face of the simplex
- Zero-sum dynamics with interior fixpoints is Hamiltonian (coexistence)
- Zero-sum dynamics with semi-definite boundary fixpoints is asymptotically Hamiltonian (competition)

From here

On this system

- ”Survival of the fittest” [AL84, p. 240] for boundary dynamics
- Degenerate replicator dynamics [HS98, p. 235]
- Dynamics on convex energy surfaces [HWZ98]
- Hamiltonian chaos and discrete replicator [SC03][PMC18][AL84][Sel91]
- Further investigate connection with Lotka-Volterra system [DFO98]

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On different systems

- Add interaction: bimatrix and polimatrix games [Hof96], [AD14]
- Investigate geometry of different dynamics: imitation, best-response, adaptive, mutator, …[HS98], [Aki79], [GP04]

References

Hassan Najafi Alishah and Pedro Duarte. “Hamiltonian evolutionary games”. In: *arXiv preprint arXiv:1404.5900* (2014).

Lars Ahlfors. *Complex Analysis (Third Edition)*. 3rd ed. McGraw-Hill ScienceEngineeringMath, 1979.

Ethan Akin. *The Geometry of Population Genetics*. 1st ed. Lecture Notes in Biomathematics 31. Springer-Verlag Berlin Heidelberg, 1979. ISBN: 978-3-540-09711-2,978-3-642-93128-4.

Ethan Akin and Viktor Losert. “Evolutionary dynamics of zero-sum games”. In: *Journal of mathematical biology* 20.3 (1984), pp. 231–258.

KH Bhaskara and K Viswanath. “Poisson algebras and Poisson manifolds, Pitman Res”. In: *Notes in Math* 174 (1988).

Pedro Duarte, Rui L Fernandes, and Waldyr M Oliva. "Dynamics of the attractor in the Lotka-Volterra equations". In: *journal of differential equations* 149.1 (1998), pp. 143–189.

Jean-Paul Dufour and Nguyen Tien Zung. *Poisson Structures and Their Normal Forms*. 1st ed. Progress in Mathematics 242. Birkhäuser Basel, 2005.

Ivar Ekeland. *Convexity methods in Hamiltonian mechanics*. Vol. 19. Springer Science and Business Media, 2012.

Vasyl V Gafiychuk and Anatoliy K Prykarpatsky. "Replicator - Mutator Evolutionary Dynamics". In: *Journal of Nonlinear Mathematical Physics* 11.3 (2004), pp. 350–360. DOI: 10.2991/jnmp.2004.11.3.6. eprint: https://doi.org/10.2991/jnmp.2004.11.3.6. URL: https://doi.org/10.2991/jnmp.2004.11.3.6.

Josef Hofbauer. "Evolutionary dynamics for bimatrix games: A Hamiltonian system?" In: *Journal of Mathematical Biology* 34.5 (May 1996), p. 675. ISSN: 1432-1416. DOI: 10.1007/BF02409754. URL: https://doi.org/10.1007/BF02409754.

Josef Hofbauer and Karl Sigmund. *Evolutionary games and population dynamics*. Cambridge University Press, 1998.

Helmut Hofer, Krzusztof Wysocki, and Eduard Zehnder. “The dynamics on three-dimensional strictly convex energy surfaces”. In: *Annals of Mathematics* (1998), pp. 197–289.

John M. Lee. *Introduction to Smooth Manifolds*. 2nd ed. Graduate Texts in Mathematics. Springer-Verlag New York, 2012.

Paulette Libermann and Charles-Michel Marle. *Symplectic Geometry and Analytical Mechanics*. 1st ed. Mathematics and Its Applications 35. Springer Netherlands, 1987.

Eckhard Meinrenken. *Introduction to Poisson Geometry*. Heidelberg University. 2017.

Robert McLachlan and Matthew Perlmutter. “Conformal Hamiltonian systems”. In: *Journal of Geometry and Physics* 39.4 (2001), pp. 276–300.

T Nagylaki. “Evolution of a finite population under gene conversion”. In: *Proceedings of the National Academy of Sciences* 80.20 (1983), pp. 6278–6281. ISSN: 0027-8424. DOI: `10.1073/pnas.80.20.6278`. eprint: `https://www.pnas.org/content/80/20/6278.full.pdf`. URL: `https://www.pnas.org/content/80/20/6278`.

Thomas Nagylaki. “Evolution of a large population under gene conversion”. In: *Proceedings of the National Academy of Sciences* 80.19 (1983), pp. 5941–5945.

Juan-Pablo Ortega and Tudor S. Ratiu. *Momentum Maps and Hamiltonian Reduction*. 1st ed. Progress in Mathematics 222. Birkhäuser Basel, 2004.

Juan-Pablo Ortega, Tudor S. Ratiu, and Rui Loja Fernandes. “The momentum map in Poisson geometry”. In: *American journal of mathematics* 131.5 (2009), pp. 1261–1310.

Varun Pandit, Archan Mukhopadhyay, and Sagar Chakraborty. “Weight of fitness deviation governs strict physical chaos in replicator dynamics”. In: *Chaos: An Interdisciplinary Journal of Nonlinear Science* 28.3 (2018), p. 033104.

Yuzuru Sato and James P Crutchfield. “Coupled replicator equations for the dynamics of learning in multiagent systems”. In: *Physical Review E* 67.1 (2003), p. 015206.

Reinhard Selten. *Game Equilibrium Models I: Evolution and Game Dynamics*. 1st ed. Springer-Verlag Berlin Heidelberg, 1991.

Karl Sigmund. *Introduction to Evolutionary Game Theory*. University of Vienna. 2011.

Peter D Taylor and Leo B Jonker. “Evolutionary stable strategies and game dynamics”. In: *Mathematical biosciences* 40.1-2 (1978), pp. 145–156.

Izu Vaisman. *Lectures on the Geometry of Poisson Manifolds*. 1st ed. Progress in Mathematics 118. Birkhäuser Basel, 1994.

Thanks