No-Regret Learning in Strategic Games:
Geometry and Dynamics

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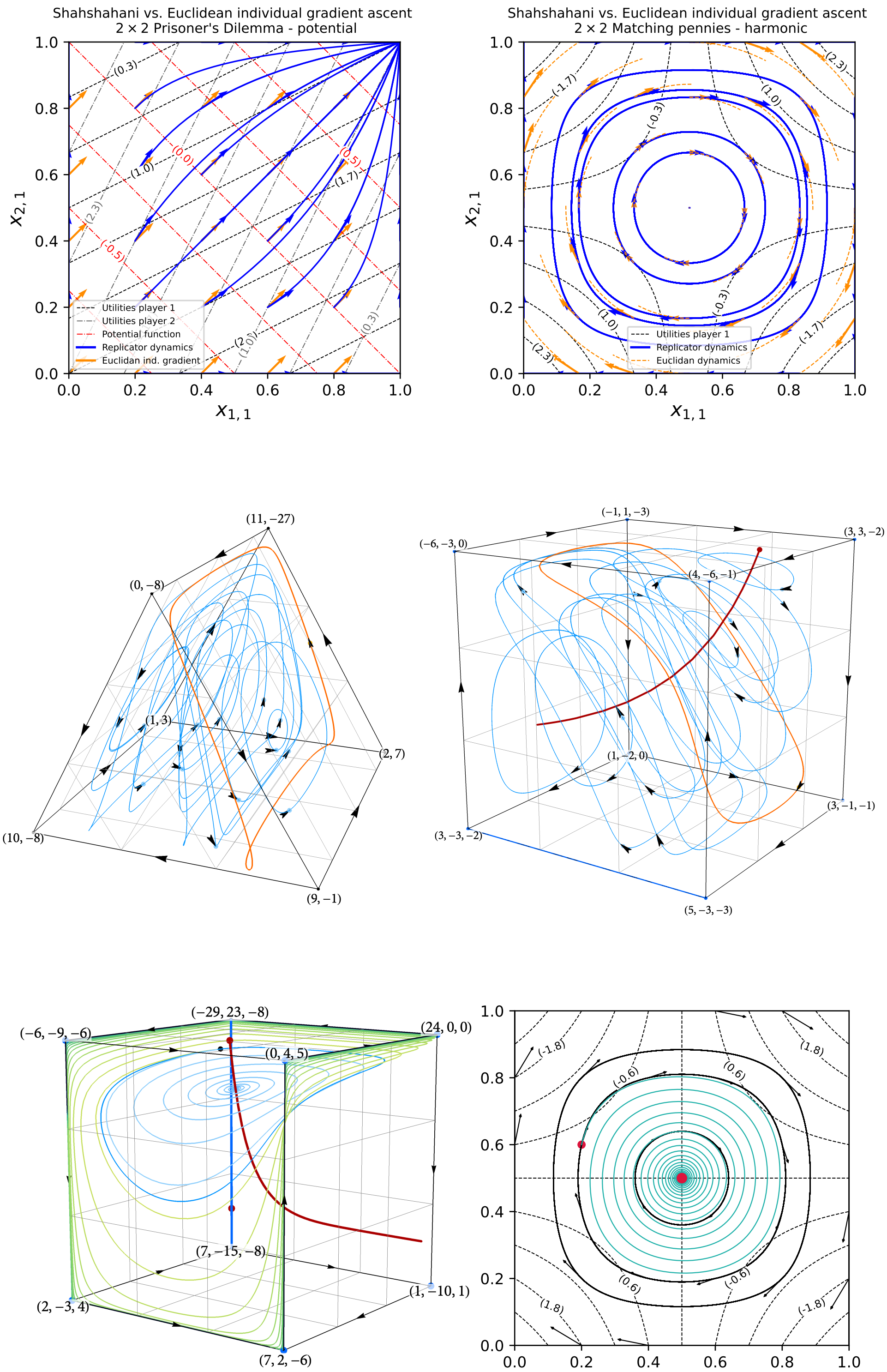
Davide Legacci   Panayotis Mertikopoulos   Bary Pradelski
Georgios Piliouras   Christos Papadimitriou



A fundamental question in multi-agent learning theory is whether players eventually learn to emulate rational behavior through repeated interactions, while minimizing their incurred regret [6, 18].

This question finds positive answer for the class of *potential games* [14], in which the players' common strategic interests give rise to a wide range of equilibrium convergence results under no-regret learning dynamics [7, 9]. By contrast, in *harmonic games* [5] – the strategic counterpart of potential games, where players have conflicting interests – very little is known outside the narrow subclass of 2-player zero-sum games with a fully-mixed equilibrium [12].

In light of this, our objectives are to 1. derive a *dynamics-driven decomposition* of a game into a potential and a harmonic component, and 2. examine the convergence properties of *follow-the-regularized-leader* (FTRL), the most widely studied class of no-regret learning schemes, in harmonic games.

We focus initially on finite games under the *replicator dynamics* (RD) of Taylor and Jonker [19], a continuous-time instance of the FTRL class with particularly appealing geometrical properties. Our starting point is Hodge-Helmholtz's theorem [4], which resolves a vector field into a potential and an incompressible component, encapsulating respectively the convergent and the non-convergent parts of the induced dynamical system. However, as we show in [11], the geometry of RD is incompatible with the Euclidean metric underpinning Helmholtz's theorem, making it necessary to consider an alternative Riemannian structure on the game's strategy space based on the so-called *Shahshahani* metric [2, 17]. In the spirit of Hodge's theorem, we then introduce the class of *Shahshahani-incompressible games* – those with vanishing Shahshahani divergence – as the opposite pole of potential games from a dynamical standpoint.

As a first result, we show that a game is Shahshahani-incompressible if and only if it is harmonic, thus establish a deep connection with a well-known decomposition for finite games into a potential and a harmonic component based on the combinatorial version of Hodge's theorem [5, 8]. Furthermore, we show that the RD on incompressible games admit a constant of motion and are volume preserving with respect to the Shahshahani metric, implying in turn that the replicator dynamics in harmonic games are Poincaré recurrent – that is, they return arbitrarily close to their starting point infinitely often, and in particular fail to converge.

We then significantly broaden the spectrum of our investigation along two axes, namely by 1. considering the class of *generalized finite harmonic games* [1], and 2. considering the whole class of FTRL dynamics, in continuous as well as discrete time. In this more general setting, we show that Poincaré recurrence persists under continuous time FTRL dynamics – including in particular RD as a special case. In discrete time, the standard implementation of FTRL may lead to even worse outcomes, spiraling towards the boundary of the game's strategy space and eventually trapping the players in a perpetual cycle of best-responses. However, if FTRL is augmented with a suitable extrapolation step – which includes as special cases the optimistic and mirror-prox variants of FTRL [13, 15] – we show that learning converges to a Nash equilibrium from any initial condition, and all players are guaranteed at most $\mathcal{O}(1)$ regret.

Finally, as part of ongoing work, we aim at leveraging Hodge's decomposition theorem for differential forms on Riemannian manifolds [20] to derive a dynamics-motivated decomposition of games with continuous action sets [16]; the primary challenge in this direction is that of constructing a suitable family of isometries for the Hessian-Riemannian metrics that underpin the FTRL dynamics [3, 10].

Since the seminal work of Candogan et al. [5], except for certain very special cases, learning dynamics in harmonic games had not been understood. This work provides an in-depth understanding of no-regret learning in harmonic games, nesting prior results on 2-player zero-sum games, and showing that potential and harmonic games are complementary to each other – not only from a strategic, but also from a dynamic viewpoint – and thereby significantly advancing a long-standing question in the field.

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