Zero-sum evolutionary games and convex Hamiltonian systems

FOUNDATIONS

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Idea

- The replicator dynamical system models the evolution of the aggregate behavior of individuals in a population.
- This system is Hamiltonian in the appropriate geometrical framework.

- Find periodic time evolutions with prescribed energy using the convexity of the Hamiltonian.

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Dynamical system on categorical probability distributions[1]

- Discrete alphabet $\mathcal{S}(n+1) \ni i$
- $p \in P(\mathcal{S}(n+1)) = \Delta^n, \quad p \mapsto x \,:\, x^i = p(i)$

$$x \in \Delta^n \subset \mathbb{R}^{n+1} = \{x \in \mathbb{R}^{n+1} : \sum_i x^i = 1, x^i \geq 0\}$$

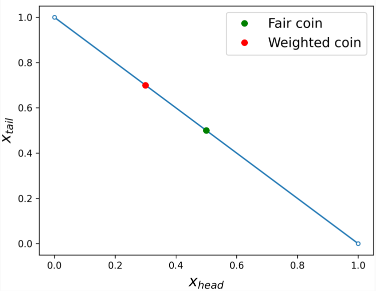

Figure 1: Space of PDs $x = (x_{head}, x_{tail})$

$$\dot{x}(t) = X_{\text{rep}}(x(t))$$

Leaves interior $\mathring{\Delta}^n$ invariant

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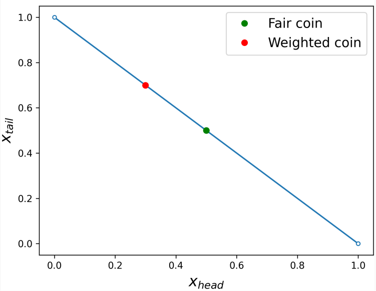

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Replicator dynamical system - Population Dynamics[2]

- Population composed of $n+1$ types or species
- The fitness or growth rate of each species $F : \mathbb{R}_+^{n+1} \to \mathbb{R}^{n+1}$ depends on the composition of the whole population

$$\dot{P}_i(t) = P_i(t)\, F_i\left(P(t)\right), \quad P \in \mathbb{R}_+^{n+1}$$

- Descend from $\mathbb{R}_+^{n+1}$ through a normalization map onto the simplex to the replicator equation, i.e. look at PD $x \in \Delta^n$ on the set of species, with $x^i = \frac{P_i}{\sum_j P_j}$

$$\dot{x}^i = x^i \left(f_i(x) - \sum_h x^h f_h(x) \right), \quad f_h(x) = F_h(P)$$

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Replicator dynamical system - Evolutionary Game Theory[3]

Population game $(\mathcal{S}(n+1), f)$: strategically interacting agents

- Agents choose a pure strategy from a finite set $\mathcal{S}(n+1)$
- The payoff of each pure strategy $f : \Delta^n \to \mathbb{R}^{n+1}$ depends on current population state $x \in P(\mathcal{S}(n+1)) = \Delta^n$

Mean dynamics via *revision protocol* $\rho : \Delta^n \to \mathbb{R}_+^{(n+1)\times(n+1)}$

$$\dot{x}^i = \left(\sum_j x^j \rho_{ji}(x)\right) - \left(x^i \sum_j \rho_{ij}(x)\right)$$

$$\rho_{ij}(x) = x^j \left(f_j(x) - f_i(x)\right)_+ \quad \text{[Imitation]}$$

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Replicator dynamical system - Information geometry[4]

Riemannian game = P.G. with Riemannian metric on $P(\mathcal{S}(n+1))$

- Gain $G(x,v) = \sum_i f_i(x)\, v^i, \quad v \in T_x\Delta, \quad f$ payoff
- Cost $C(x,v) = \frac{1}{2}\, \|v\|_x^2$

$$\dot{x} = \arg\max_{v \in T_x\Delta} \left(G(x,v) - C(x,v)\right)$$

- Replicator with *Fisher-Shahshahani metric* $g_{ij}(x) = \delta_{ij} \,/\, x^i$
- Replicator fields ⊃ Fisher gradients
  - E.g. **linear symmetric payoff** replicator field
  - Wright and Fisher, classical population genetics

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Zero-sum replicator systems

- Average payoff vanishes identically $\sum_i x^i f_i(x) \equiv 0$
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- Extensively studied in classical GT[5]
- Very restrictive assumption for real life applications
- *Discrete* zero-sum replicator: model for gene conversion[6]
- Interesting in its own right for Hamiltonian character

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Poisson structure on a space M

- General framework: *stratified space M*

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M) \quad \text{[A.S., Leibnitz, Jacobi]}$$
$$\{f, g\} = \{x^i, x^j\}\, \partial_i f\, \partial_j g = \pi^{ij}\, \partial_i f\, \partial_j g$$

- π: $\binom{2}{0}$ tensor-field [Anti-symmetric, Jacobi]
- Hamiltonian vector fields and dynamical systems

$$X_H = \pi(\mathrm{d}H, \cdot) \qquad X_H^i = \pi^{hi}\, \partial_h H$$
$$\dot{x} = X_H(x) \qquad \dot{x}^i = \{H, x^i\}$$

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Poisson structure on a space M - degeneracy

- π, and equivalently $\{\cdot, \cdot\}$, can be degenerate
- No restriction on the dimension of M

$$M = \mathbb{R}^3, \{x^i, x^j\} = \begin{pmatrix} 0 & 1 & A \\ -1 & 0 & B \\ -A & -B & 0 \end{pmatrix}$$

- Casimir $f(x) = Bx^1 - Ax^2 + x^3$, namely $\{f, \cdot\} \equiv 0$
- Change coordinates to isolate degeneracy

$$y^1 = x^1, y^2 = x^2, y^3 = Bx^1 - Ax^2 + x^3$$
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Stratified Poisson structure for the standard simplex[7]

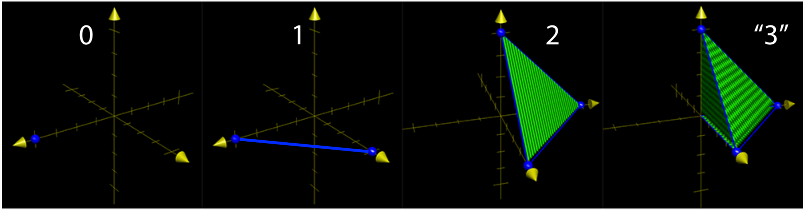

Figure 2: Simplices representable in three dimensions. Each face is a Poisson manifold.

Poisson structure on Δ^n with A anti-symmetric $(n+1)$ matrix

$$\{x^i, x^j\}_A = x^i x^j \left(\sum_h (A_{ih} + A_{hj}) x^h - A_{ij} \right)$$

[7] Regular and Singular Poisson Reduction Theorems [OR04, p. 364] [ORF09, p. 1273]

Interior Hamiltonian dynamics of zero-sum replicator

- Interior fixpoint $q \in \mathring{\Delta}^n$
- $H_q(x) = D_{KL}(q \| x) = \sum_i q^i \log \frac{q^i}{x^i}$ Relative entropy
  - Provides the Fisher metric
  - Appears in EGT as Lyapunov function given ESS strategy

Theorem

Consider a replicator dynamical system with anti-symmetric payoff matrix A. If a fixpoint q exists in $\mathring{\Delta}^n$, then the system is Hamiltonian w.r.t. $\{x^i, x^j\}_A$, with $H_q(x)$ as Hamiltonian function[8].

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Interior Hamiltonian dynamics of zero-sum replicator

- Interior trajectories do not converge to the boundary nor to a fixpoint
- Bounded orbits, periodic or not?

Interior Hamiltonian dynamics of zero-sum replicator

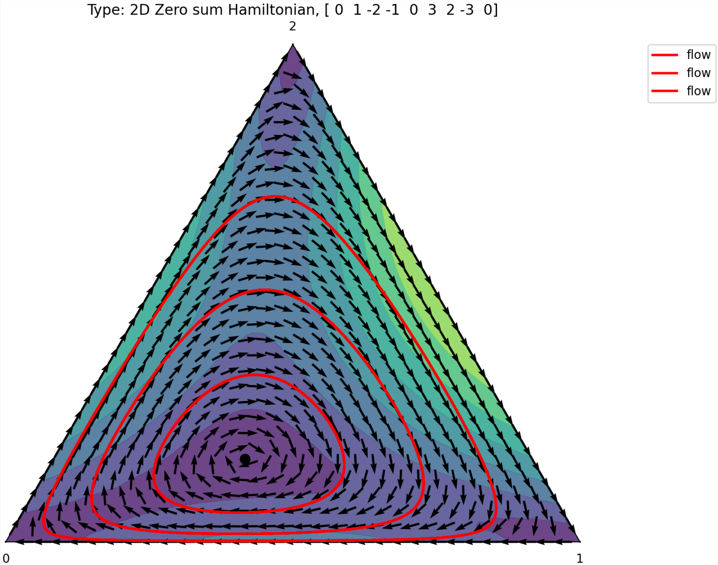

Figure 3: Three periodic orbits around the fixpoint.

Interior Hamiltonian dynamics of zero-sum replicator

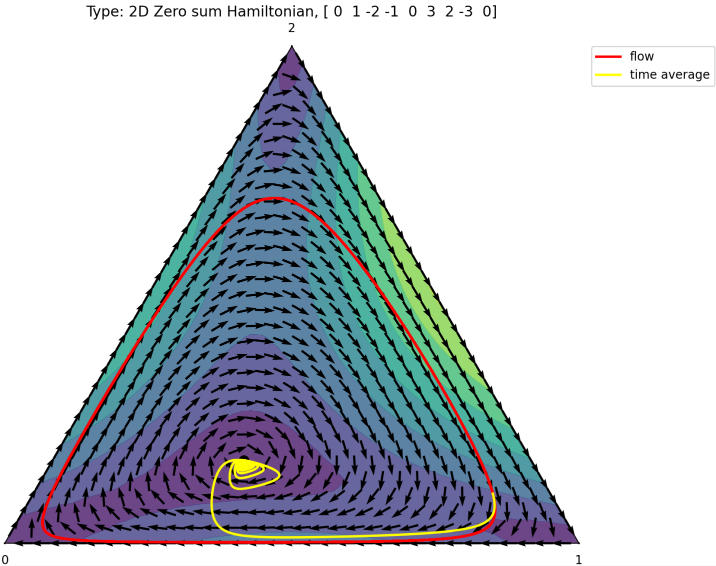

Figure 4: One periodic orbit. The time average converges to the fixpoint.

Interior Hamiltonian dynamics of zero-sum replicator

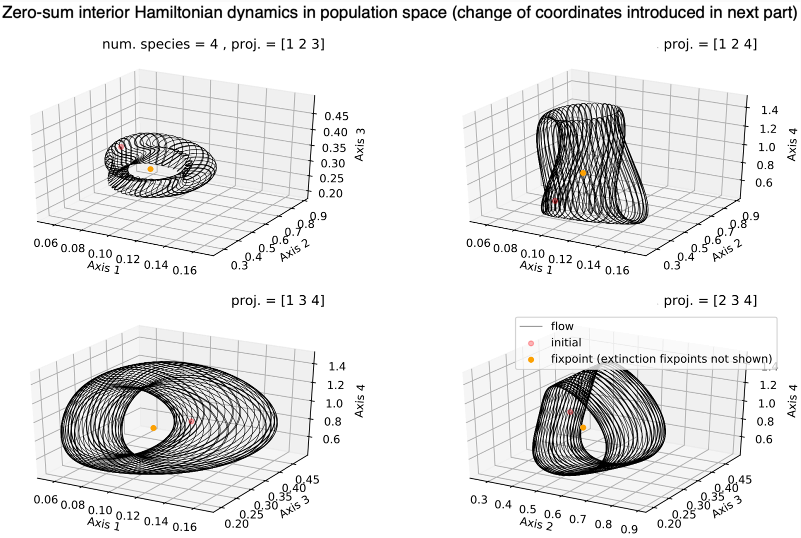

Figure 5: Non periodic bounded orbits

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Thanks