Decomposition of Normal Form Games

Harmonic, Potential, and Non-Strategic Games

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Mission

Goal

- Identify classes of games with distinctive strategic and dynamical properties

Plan

- Represent a finite normal form game as a graph
- Use this representation to understand the decomposition the space of games into three components
- Study the properties of these components

Starting point

- Candogan et al. 2011

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The Response Graph of a Normal Form Game

Definitions

A normal form game is a tuple $\Gamma = (\mathcal{N}, \mathcal{A}, u)$ where

- $\mathcal{N} = \{1, 2, \ldots, N\}$ is the set of **players**
- Each player $i \in \mathcal{N}$ has a set of **pure strategies**

$$\mathcal{A}_i = \{1, 2, \ldots, A_i\}$$

- $\mathcal{A} = \prod_{i \in \mathcal{N}} \mathcal{A}_i$ is the set of **pure strategy profiles**
- Each player has an individual utility function

$$u_i : \mathcal{A} \to \mathbb{R}, \quad a \mapsto u_i(a)$$

- The **utility map of the game** is

$$u : \mathcal{A} \to \mathbb{R}^N, \quad a \mapsto (u_1, \ldots, u_N)(a)$$

Definitions

Given the normal form game $\Gamma = (\mathcal{N}, \mathcal{A}, u)$

- The number of players is

$$N = |\mathcal{N}|$$

- The number of pure strategies of player $i \in \mathcal{N}$ is

$$A_i = |\mathcal{A}_i|$$

- The number of pure strategies profiles is

$$A = |\mathcal{A}| = \prod_{i \in \mathcal{N}} A_i$$

$\Rightarrow$ the number of utilities is AN

Example - 2×3 normal form game

- $\mathcal{N} = \{1, 2\}$
- $\mathcal{A}_1 = \{1, 2\}$, $\mathcal{A}_2 = \{1, 2, 3\}$
- $\mathcal{A} = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$
- $AN = 12$

$$u : \mathcal{A} \to \mathbb{R}^2$$

$$\begin{aligned}
(1,1) &\longmapsto (-3,3) \\
(1,2) &\longmapsto (0,-5) \\
(1,3) &\longmapsto (-3,3) \\
(2,1) &\longmapsto (3,0) \\
(2,2) &\longmapsto (-3,0) \\
(2,3) &\longmapsto (0,1)
\end{aligned}
\quad \xleftrightarrow{\text{bimatrix notation}} \quad
\begin{pmatrix} -3,3 & 0,-5 & -3,3 \\ 3,0 & -3,0 & 0,1 \end{pmatrix}$$

Vector Space of Utilities

Given a set of players $\mathcal{N}$ and a set of pure strategy profiles $\mathcal{A}$

- A utility map $u : \mathcal{A} \to \mathbb{R}^N$ is the assignment of N numbers to each of the A strategy profiles
- Denote the space of utilities by $\mathcal{U}$
- $\mathcal{U}$ is an AN-dimensional vector space

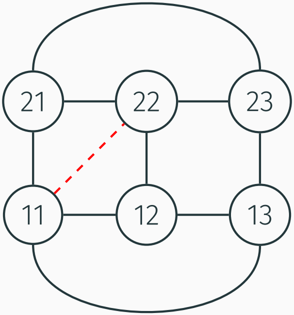
Example - 2×3 game

$$u = \begin{pmatrix} u_1(1,1) \\ u_1(1,2) \\ \vdots \\ u_2(2,2) \\ u_2(2,3) \end{pmatrix} \in \mathcal{U}, \quad \dim \mathcal{U} = 12$$

Response Graph

Let's build a graph from a normal form game $(\mathcal{N}, \mathcal{A}, \cdot)$

- Draw a node for each pure strategy profile in $\mathcal{A}$
- Draw an edge between strategy profiles that differ only in the strategy of one player



Edges, Unilateral Deviations, Actor

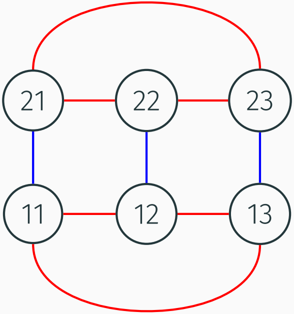
- Pairs of strategy profiles $a \in \mathcal{A}, b \in \mathcal{A}$ that differ only in the strategy of one player are called unilateral deviations
- Their space - that is the space of edges of the response graph - is denoted by $\mathcal{E}$

$$E = |\mathcal{E}| = \frac{A}{2} \sum_{i \in \mathcal{N}} (A_i - 1)$$

- For each edge, the player who is deviating is called the actor of the deviation

$$\begin{aligned} \mathrm{act} : \mathcal{E} &\rightarrow \mathcal{N} \\ (ab) &\longmapsto i \text{ such that } a_i \neq b_i \end{aligned}$$

Example 2×3 - Edges, Unilateral Deviations, Actor



$$\text{act}(\text{blue edges}) = 1$$

$$\text{act}(\text{red edges}) = 2$$

Utilities, Flows, and Deviations

We built the response graph just with $(\mathcal{N}, \mathcal{A}, \cdot)$.

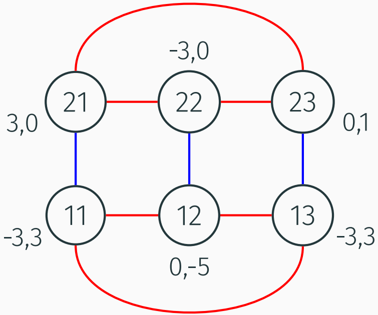
Let's now add the utilities to the picture.

Goal - build

$$\text{Deviation Map} : \text{Utilities Space} \to \text{Flows Space}$$

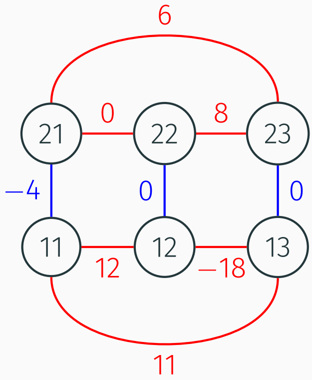
Recall - Vector Space of Utilities

- A utility $u : \mathcal{A} \to \mathbb{R}^N$ is the assignment of N numbers to each of the A nodes of the response graph
- Denote the space of utilities by $\mathcal{U}$
- $\mathcal{U}$ is an AN-dimensional vector space



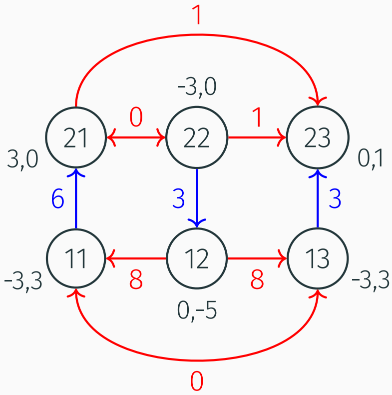
Vector Space of Flows

- A flow $X : \mathcal{E} \rightarrow \mathbb{R}$ is the assignment of one number to each of the E edges of the response graph
- Denote the space of flows by $\mathcal{F}$
- $\mathcal{F}$ is an E-dimensional vector space



Build a special flow for the normal form game $(\mathcal{N}, \mathcal{A}, u)$

- Assign to each edge the actor's utilities difference
- Call this flow deviation flow of the game



- We assign the number $u_i(b) - u_i(a)$ with $i = \mathrm{act}(ab)$ to the edge $a \to b$
- Always choose the orientation such that this number is ≥ 0
- If an arrow *leaves* a node a player following the arrow *does not lose*

The Deviation Map

- Take a normal form game $(\mathcal{N}, \mathcal{A}, u)$
- $\mathcal{U}$ is the utilities vector space
- $\mathcal{F}$ is the flows vector space
- Map the utility of the game to its deviation flow

Definition

$$\begin{aligned} D : \mathcal{U} &\to \mathcal{F} \\ u &\longmapsto Du \end{aligned} \quad \textit{such that} \quad \begin{aligned} Du : \mathcal{E} &\to \mathbb{R} \\ (ab) &\longmapsto u_i(b) - u_i(a) \\ &\textit{for } i = act(ab) \end{aligned} \tag{DM}$$

This map is linear, and is called deviation map.

Why the deviation map $D : \mathcal{U} \to \mathcal{F}$ is useful

The deviation flow of a game Du captures its strategic structure

- Loosely speaking, the strategic structure of a game is the orientation of the edges of its response graph
- It captures the interest of each player at each state (strategy profile) of the game

- Games with different utilities u, u' may have the same strategic structure
- This happens in particular if they have the same deviation flow, that is if $Du = Du'$

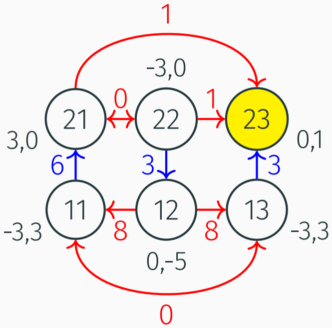
Example - Pure Nash Equilibrium (NE)

A Pure Nash Equilibrium for a game $(\mathcal{N}, \mathcal{A}, u)$ is a strategy profile $a \in \mathcal{A}$ such that

$$u_i(a) \geq u_i(b) \text{ for all } b \in \mathcal{A} \text{ such that } (ab) \in \mathcal{E}_i, \text{ for all } i \in \mathcal{N}$$

The deviation flow Du fully determines the set of NE

$$Du(ba) \geq 0 \text{ for all } b \in \mathcal{A} \text{ such that } (ab) \in \mathcal{E}$$



Utility Space Decomposition

Utilities Space Decomposition

Goal - Introduce the decomposition of the utilities space $\mathcal{U}$ into the three components

$$\mathcal{U} = \mathcal{K} \oplus \mathcal{P} \oplus \mathcal{H}$$

These components are determined by deviation flows:

$$\mathcal{K}, \mathcal{P}, \mathcal{H} = \{u \in \mathcal{U} : Du \text{ fulfills some property}\}$$

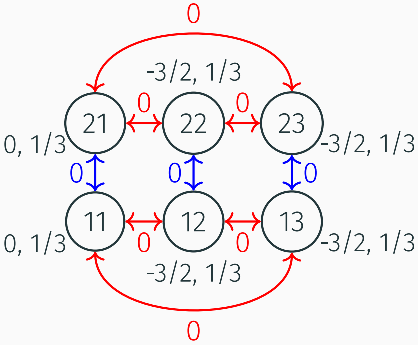
and are easy to visualize on a response graph.

1. Definition of the components
2. Statement of the decomposition theorem
3. Sketch one crucial step of the proof (original)

Definition

The non-strategic component of $\mathcal{U}$ is the subspace of utilities with vanishing deviation flow

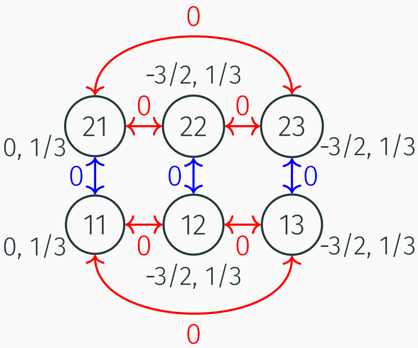
$$\boxed{\mathcal{K} \coloneqq \{u \in \mathcal{U} : Du = 0\}} \tag{$\mathcal{K}$}$$



Why “Non-Strategic”?

- A game $(\mathcal{N}, \mathcal{A}, u)$ with $u \in \mathcal{K} \subseteq \mathcal{U}$ is called non-strategic
- The deviation flow of a non-strategic game is identically zero

In a non-strategic game all players are indifferent between all of their strategies since no deviation will lead to any gain



Recall - Complement and Direct Sum

A complement $\overline{S}$ of a subspace $S \subseteq V$ is a subspace $\overline{S}$ of V s.t.

- Any $v \in V$ can be written as the sum of some $s \in S$ and $\bar{s} \in \overline{S}$

$$v = s + \bar{s}$$

- $S \cap \overline{S} = \{0\}$

If $\overline{S}$ is a complement of S we say that $\overline{S}$ and S are in direct sum:

$$S \oplus \overline{S} = V$$

Any $S \subseteq V$ admits a complement, that in general is not unique

Choose a Complement of the Non-Strategic Component

A Normalization is a *choice* of a complement $\overline{\mathcal{K}}$

By definition

$$\mathcal{U} = \mathcal{K} \oplus \overline{\mathcal{K}}$$

Recall - Given a utility u, its deviation flow Du captures its strategic structure.

If $u \in \mathcal{U}$ then $u = u_{\mathcal{K}} + \overline{u}$ for some $u_{\mathcal{K}} \in \mathcal{K}$ and $\overline{u} \in \overline{U}$. So

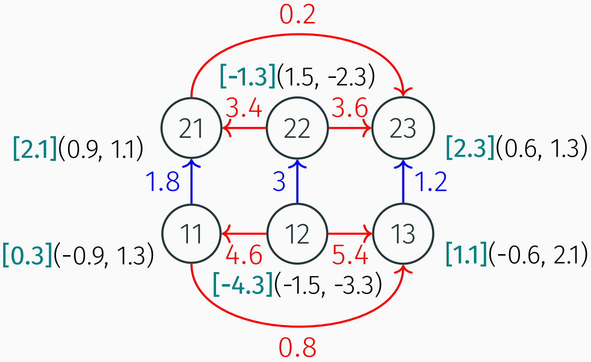
$$Du = Du_{\mathcal{K}} + D\overline{u} = D\overline{u}$$

By looking at the normalized component of the utility function we retain all of the strategic structure of the game.

Potential Games

A normal form game $(\mathcal{N}, \mathcal{A}, u)$ is called potential if there exists a function $\phi : \mathcal{A} \to \mathbb{R}$ such that the deviation flow Du is

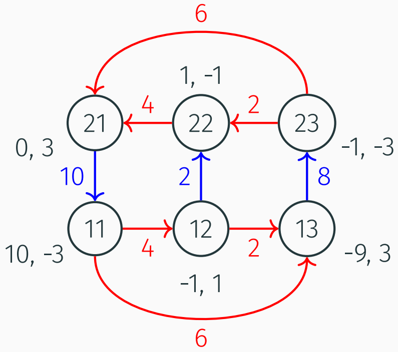
$$Du(ab) = \phi(b) - \phi(a) \quad \text{for each } (ab) \in \mathcal{E} \qquad \text{(potential)}$$



Harmonic Games

A normal form game $(\mathcal{N}, \mathcal{A}, u)$ is called harmonic if the net deviation flow at each node of the response graph is zero:

$$\sum_{b:(ab)\in\mathcal{E}} Du(ab) = 0 \quad \text{for each } a \in \mathcal{A} \qquad \text{(harmonic)}$$



- May know the *graph Laplacian* $\Delta_0 =$ degree M. $-$ adjacency M.
- Generalize to *vector graph Laplacian* $\Delta_1 : \mathcal{F} \to \mathcal{F}$
- Harmonic flows annihilate Δ_1
- Inner product dependent - Euclidean
- Good introduction: Lim 2020

Normalization: $\mathcal{P}$ and $\mathcal{H}$

A generic potential/harmonic game is not normalized, i.e. given a potential/harmonic game nothing forbids that $Du = 0$

Definition

$\mathcal{P}$ is the subspace of normalized potential games

$$\boxed{\mathcal{P} := \{u \in \mathcal{U} : u \text{ is potential}\} \cap \overline{K}} \tag{$\mathcal{P}$}$$

Definition

$\mathcal{H}$ is the subspace of normalized harmonic games

$$\boxed{\mathcal{H} := \{u \in \mathcal{U} : u \text{ is harmonic}\} \cap \overline{K}} \tag{$\mathcal{H}$}$$

Visualize it!

$\mathcal{U} = \mathcal{K} \oplus \mathcal{P} \oplus \mathcal{H}$

Theorem (Candogan et al. 2011)

Fixed $\mathcal{N}$, $\mathcal{A}$, and a choice of normalization $\bar{\mathcal{K}}$, the space of utilities $\mathcal{U}$ decomposes uniquely as

$$\mathcal{U} = \mathcal{K} \oplus \mathcal{P} \oplus \mathcal{H}$$

Any game $(\mathcal{N}, \mathcal{A}, u)$ admits a unique decomposition into

- *A non-strategic game $(\mathcal{N}, \mathcal{A}, u_{\mathcal{K}})$*
- *A normalized potential game $(\mathcal{N}, \mathcal{A}, u_{\mathcal{P}})$*
- *A normalized harmonic game $(\mathcal{N}, \mathcal{A}, u_{\mathcal{H}})$*

with

$$u = u_{\mathcal{K}} + u_{\mathcal{P}} + u_{\mathcal{H}}$$

Hand-wavy Explanation: Helmholtz Decomposition

Any "regular" vector field in three dimensions can be decomposed into the sum of

- a gradient field, that is curl-free (or *irrotational*)
- a curl field, that is divergence-free (or *solenoidal*)

$$\vec{X} = \vec{\nabla}\phi + \vec{\nabla}\times\vec{A}$$

(E.g. electromagnetic field)

This is analogue to the decomposition of the normalized utility into potential and harmonic components:

- gradient field $\sim$ potential component
- divergence-free field $\sim$ harmonic component

Our First Result: Alternative Proof

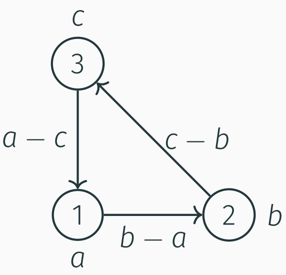
- The proof by Candogan et al. 2011 heavily relies on an explicit choice of normalization
- We developed a proof that does not depend on such choice

1. $\mathcal{U} = \mathcal{K} \oplus \overline{\mathcal{K}} \cong \mathcal{K} \oplus \operatorname{Im} D$ - standard
2. $\operatorname{Im} D \cong \operatorname{Im} d_0 \oplus {}^{\operatorname{Im} D}/_{\operatorname{Im} d_0}$ - standard
3. $\operatorname{Im} D = \ker d_1$ - nontrivial, original
4. ${}^{\ker d_1}/_{\operatorname{Im} d_0} \cong \ker \Delta_1$ - Hodge theorem

$$\boxed{\mathcal{U} \cong \mathcal{K} \oplus \operatorname{Im} d_0 \oplus \ker \Delta_1} \tag{1}$$

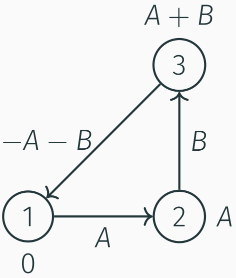
3. Deviation flows are precisely closed: $X(ab)+X(bc)+X(ca) = 0$

$\operatorname{Im} D \subseteq \ker d_1$



$$(d_1 Du)(abc) =$$
$$= Du(ab) + Du(bc) + Du(ca)$$
$$= b - a + c - b + a - c = 0$$

$\operatorname{Im} D \supseteq \ker d_1$



$$0 = (d_1 X)(abc) =$$
$$= X(ab) + X(bc) + X(ca)$$
$$\Rightarrow \exists u : Du = X$$

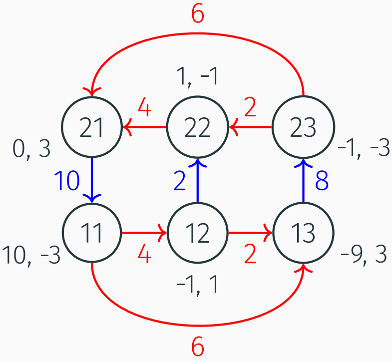
Higher order: response graph factorization and Poincarè lemma

Properties of the Components

Theorem (Candogan et al. 2011)

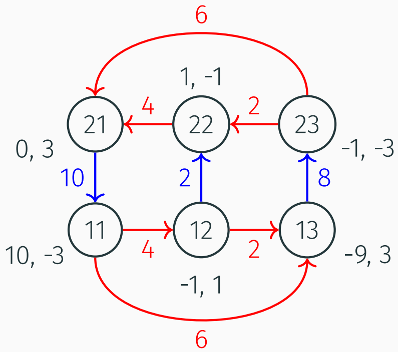
Harmonic games generically do not have pure NE.

Intuition.
The net flow at each node is zero, so generically no node has only incoming arrows. □



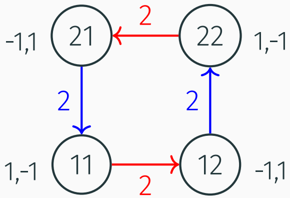
Theorem (Candogan et al. 2011)

Harmonic normal form games always admit the uniformly mixed strategy profile as mixed NE.



$$x^* = \left(\frac{1}{2}, \frac{1}{2}\right) \times \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Matching Pennies

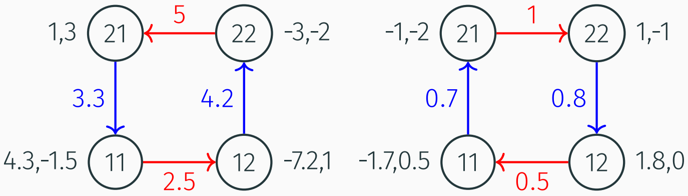


$$x^* = \left(\frac{1}{2}, \frac{1}{2}\right) \times \left(\frac{1}{2}, \frac{1}{2}\right)$$

Our Second Result: Harmonic Games and Mixed Nash Revisited

- The proof by Candogan et al. 2011 relies on the use of the Euclidean inner product
- We generalized the notion of harmonic games considering non-Euclidean inner products

Work result A 2x2 strategic normal form game that is harmonic with respect to a diagonal inner product admits a fully mixed NE that depends only on the inner product, and not on utilities.



$$x^* = \left(\frac{2}{3}, \frac{1}{3}\right) \times (0.56, 0.44)$$

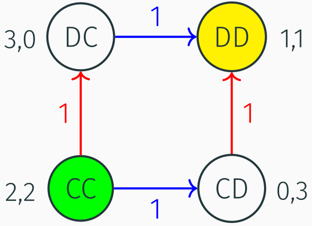
Time Check

- ⏩ Properties of Non-Strategic and Potential Games
- ⏩ Conclusions

Non-Strategic Component: Pareto Efficiency

A pure strategy profile $a \in \mathcal{A}$ is Pareto efficient if it is impossible to make one player better off without making another player worse off.

$$a \in \mathcal{A} \text{ is PO} \iff \nexists\, b \in \mathcal{A} : \begin{cases} u_i(b) \geq u_i(a) & \text{for all } i \in \mathcal{N} \\ u_j(b) > u_j(a) & \text{for some } j \in \mathcal{N} \end{cases}$$



The Non-Strategic Component Affects Efficiency

- Consider two games whose difference is non-strategic:

$$u - v \in \mathcal{K} \Leftrightarrow Du = Dv \quad \text{same strategic structure}$$

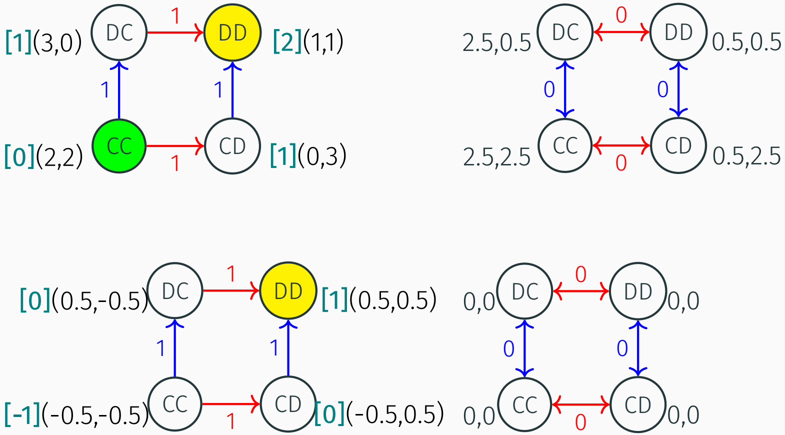
- What changes is the equilibria efficiency

Theorem (Candogan et al. 2011)

For any normal form game $(\mathcal{N}, \mathcal{A}, u)$ there exists a normal form game $(\mathcal{N}, \mathcal{A}, v)$ such that

- *The difference between u and v is non-strategic*
- *The sets of pure Nash equilibria and of Pareto efficient strategies of $(\mathcal{N}, \mathcal{A}, v)$ coincide*

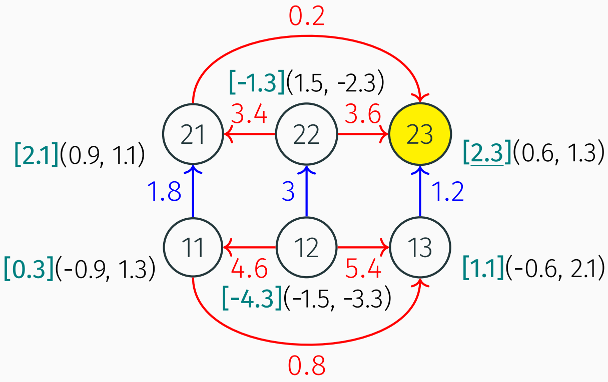
Example - Prisoner's Dilemma $u = u_{\mathcal{K}} + u_{\mathcal{P}} + u_{\mathcal{H}}$



Theorem (Monderer and Shapley 1996)

Every potential normal form game has at leas one pure NE.

Proof.
The potential function $\phi : \mathcal{A} \to \mathbb{R}$ always has a maximum in $\mathcal{A}$, so each argmax of ϕ is a pure Nash equilibrium. □



Conclusions and Open Directions

Conclusions And Open Directions

Non-Strategic, Potential and Harmonic games display distinctive properties that depend on some explicit choices.

Results so far

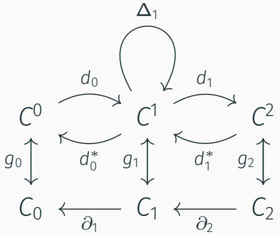
- Normalization-independent proof of decomposition theorem
- Result on mixed NE of non-Euclidean harmonic games

Research directions

- Dynamical (Balduzzi et al. 2018, Letcher et al. 2019)
  - Behavior of player dynamics in harmonic games
  - Decomposition of dynamics vector field
- Strategic (Abdou et al. 2020)
  - Behavior of the decomposition under strategic transformations

Remark - The elephant in the room

I tried to describe the procedure from a game theoretical point of view.
The proof of the decomposition theorem actually relies on the rich machinery of **simplicial cohomology** and **combinatorial Hodge theory**.
If you're curious, get in touch.



$$\Delta_1 = d_0 \circ d_0^* + d_1^* \circ d_1$$

$$\text{exact} := \operatorname{Im} d_0$$

$$\text{closed} := \ker d_1$$

$$\text{harmonic} := \ker \Delta_1$$

$$\begin{aligned} C^1 &= \operatorname{Im} d_0 \oplus \operatorname{Im} d_1^* \oplus \ker \Delta_1 \\ &= \text{exact} \oplus (\text{closed})^\perp \oplus \text{harmonic} \end{aligned}$$

$$^{\text{closed}}/_{\text{exact}} \cong \text{harmonic}$$

Thank You

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Proof Sketch

Our First Result: Alternative Proof

- The proof by Candogan et al. 2011 heavily relies on an explicit choice of normalization
- We developed a proof that does not depend on such choice

1. $\mathcal{U} = \mathcal{K} \oplus \overline{\mathcal{K}} \cong \mathcal{K} \oplus \operatorname{Im} D$ - standard
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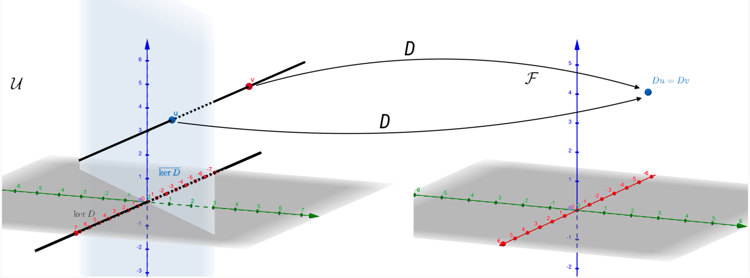
$$\boxed{\mathcal{U} \cong \mathcal{K} \oplus \operatorname{Im} d_0 \oplus \ker \Delta_1} \tag{2}$$

1. Proof of $D : \overline{\mathcal{K}} \cong \operatorname{Im} D$



$$D : \overline{\mathcal{K}} \cong \operatorname{Im} D$$

Let $u, v \in \overline{\mathcal{K}}$. If $Du = Dv$ then $u - v \in \mathcal{K} \cap \overline{\mathcal{K}} = \{0\}$ $\square$

Let $w \in \operatorname{Im} D$. Then $w = Du = D(u' + k) = Du'$ with $u' \in \overline{K}, k \in K$ $\square$

3. Proof of $\operatorname{Im} D = \ker d_1$ - Step (i)

The fact that $\operatorname{Im} D \subseteq \ker d_1$ is stated in Candogan et al. 2011, but it is proved employing a relatively heavy machinery, while we developed a simpler argument:

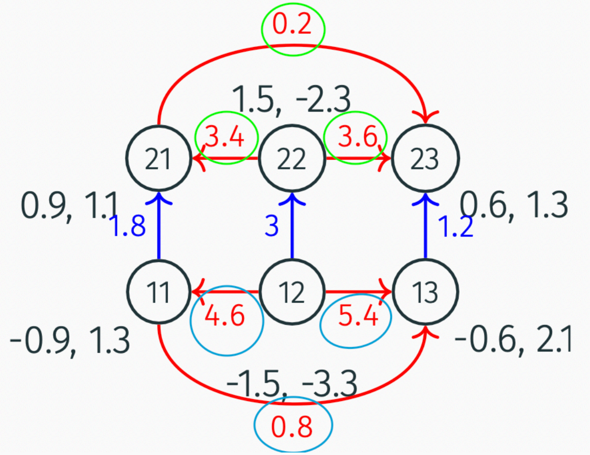
$$\begin{aligned}(d_1 D u)(abc) &= Du_{ab} + Du_{bc} + Du_{ca} \\ &= u_i(b) - u_i(a) + u_j(c) - u_j(b) + u_h(a) - u_h(c) \\ &= 0 \text{ since } (abc) \text{ is a 3-clique} \Rightarrow i = j = h \quad \square\end{aligned}$$

This means that $d_1 \circ D \equiv 0$, i.e. every deviation flow is a closed flow. Note that being a deviation flow is in spirit analogue to being exact, since D is in spirit a generalization of d_0.

Visualize $\operatorname{Im} D \subseteq \ker d_1$

The net flow over any 3-clique is zero

$$(d_1 Du)(abc) = Du(ab) + Du(bc) + Du(ca) = 0$$



3. Proof of $\operatorname{Im} D = \ker d_1$ - Step (ii)

The proof of the fact that $\operatorname{Im} D \supseteq \ker d_1$ is, to our knowledge, original. The statement is that every closed flow is the deviation flow of some game.

Given a closed flow X we need to find a utility u such that $Du = X$. The idea is to factorize the response graph into complete sub-graphs that have a unique actor, and to decouple the system of equations $Du(ab) = X(ab)$ into sub-systems relative to these sub-graphs. With this decomposition in place the problem is reduced to showing that if X is closed than it is exact *on each complete sub-graph*. This is true by Poincarè lemma since each complete sub-graph is contractible[1].

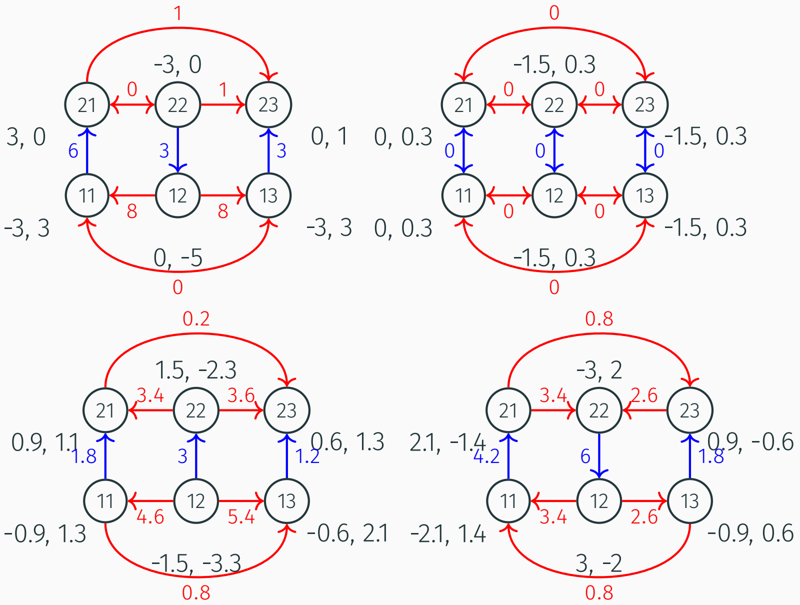
[1] "3-cliques are full": as a 2-dimensional simplicial complex, 3-cliques generate the space of 2-chains.

Proof sketch - Conclusion

1. $\mathcal{U} = \mathcal{K} \oplus \overline{\mathcal{K}} \cong \mathcal{K} \oplus \operatorname{Im} D$ - standard
2. $\operatorname{Im} D \cong \operatorname{Im} d_0 \oplus {}^{\operatorname{Im} D}/_{\operatorname{Im} d_0}$ - standard
3. $\operatorname{Im} D = \ker d_1$ - From previous slides
4. ${}^{\ker d_1}/_{\operatorname{Im} d_0} \cong \ker \Delta_1$ - Hodge theorem

$$\mathcal{U} \cong \mathcal{K} \oplus \underbrace{\overbrace{\operatorname{Im} d_0}^{\text{potential}} \oplus \overbrace{\ker \Delta_1}^{\text{harmonic}}}_{\operatorname{Im} D}$$

Drafts


1
-3, 0
0
1
21
22
23
3, 0
6
3
3
0, 1
11
12
13
8
8
-3, 3
0, -5
-3, 3
0
0
-1.5, 0.3
0
0
21
22
23
0, 0.3
0
0
0
-1.5, 0.3
11
12
13
0
0
0, 0.3
-1.5, 0.3
-1.5, 0.3
0
0.2
1.5, -2.3
3.4
3.6
21
22
23
0.9, 1.1
1.8
3
1.2
0.6, 1.3
11
12
13
4.6
5.4
-0.9, 1.3
-1.5, -3.3
-0.6, 2.1
0.8
0.8
-3, 2
3.4
2.6
21
22
23
2.1, -1.4
4.2
6
1.8
0.9, -0.6
11
12
13
3.4
2.6
-2.1, 1.4
3, -2
-0.9, 0.6
0.8

Mixed Extension of a Normal Form Game

A mixed strategy for player $i \in \mathcal{N}$ is a probability distribution over the set of pure strategies $\mathcal{A}_i$

$$\text{for each } i \in \mathcal{N}, \quad x_i \in \Delta(\mathcal{A}_i) \quad \text{i.e.} \begin{cases} x_{i,a_i} \geq 0 \quad \forall a_i \in \mathcal{A}_i \\ \sum_{a_i \in \mathcal{A}_i} x_{i,a_i} = 1 \end{cases}$$

The extended payoff of player $i \in \mathcal{N}$ is the expectation value of $u_i : \mathcal{A} \to \mathbb{R}$ with respect to the product probability distribution $\mathrm{P}_x : \mathcal{A} \to \mathbb{R}$ induced by a mixed strategy profile $(x_1, \ldots, x_N)$:

$$\begin{aligned} \bar{u}_i : \prod_{i \in \mathcal{N}} \Delta(\mathcal{A}_i) &\longrightarrow \mathbb{R} \\ \underbrace{(x_1, \ldots, x_N)}_{\text{mixed strategy profile}} &\longmapsto \mathbb{E}_{a \sim x}[u_i(a)] = \sum_{a \in \mathcal{A}} u_i(a) \underbrace{\prod_{j \in \mathcal{N}} x_{j,a_j}}_{\mathrm{P}_x(a)} \end{aligned}$$

Mixed Nash Equilibrium

Analogously to a pure NE, a Mixed Nash Equilibrium for the mixed extension of a normal form game $(\mathcal{N}, \mathcal{A}, \bar{u})$ is a mixed strategy profile $(x_1, \dots, x_N)$ at which no player has interest in making a mixed unilateral deviation:

$$\bar{u}_i(x_i; x_{-i}) \geq \bar{u}_i(y_i; x_{-i}) \quad \forall\, y_i \in \Delta(\mathcal{A}_i), \quad \forall\, i \in \mathcal{N}$$

Compare with the definition of pure NE:

$$u_i(a_i; a_{-i}) \geq u_i(b_i; a_{-i}) \quad \forall\, b_i \in \mathcal{A}_i, \quad \forall\, i \in \mathcal{N}$$

Vector Space of Individual Utilities

Given a set of players $\mathcal{N}$ and a set of pure strategy profiles $\mathcal{A}$

- An individual utility $u_i : \mathcal{A} \to \mathbb{R}$ is the assignment of one number to each of the A strategy profiles
- Denote the space of individual utilities by $\mathcal{V}$
- $\mathcal{V}$ is an A-dimensional vector space

Example - 2×3 game: $N = 2, A = 6$

$$u_1 = \begin{pmatrix} u_1(1,1) \\ u_1(1,2) \\ \vdots \\ u_1(2,2) \\ u_1(2,3) \end{pmatrix} \in \mathcal{U}, \quad \dim \mathcal{V} = 6$$

The graph Laplacian acts on this space $\Delta_0 : \mathcal{V} \to \mathcal{V}$; this is C^0 in simplicial cohomology notation.