From Classical to Evolutionary Game Theory

A short survey with an eye on convexity

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1 Classical Game Theory

Von Neumann and Morgenstern (1944) elaborate *Classical Game Theory* in 1944 as a *mathematical method for economy* (Von Neumann actually lays down the foundations of game theory in 1928).

In Chapter II they formulate (both intuitively and axiomatically) the concept of *Normal Form Game* to model a situation where players simultaneously pick a strategy from a set, and each player assigns a utility value to every possible combination of the strategies.

The focus is on *zero-sum two-person games* (Chapter III), milestone model upon which more general models rely[1].

- Of interest may be Section 16, *Linearity and Convexity*.
- Leyton-Brown and Shoham (2008) give a good, modern and extremely concise introduction to the field.

1.1 Solution concepts

Most Game Theory literature is concerned with **solution concepts**, namely with how rational players can select their strategy to maximize their utility, given that each of their opponents is attempting to do the same thing. A lot of this analysis is concerned with the *classification* and *existence* of solution strategies.

- Nash (1950) (proceeding) and **Nash (1951)** (details) comes up with one of the most important solution concepts, namely that of *Nash equilibrium*.

[1] An n-person game can be reduced to a zero-sum $(n+1)$-person game, Von Neumann and Morgenstern 1944, p. 505

- Symmetric games always admit a Nash equilibrium (Nash's proof is based on Brouwer's fixed point theorem).
- In the two-person zero-sum case the *main theorem* of Von Neumann and Morgenstern and the existence of a Nash equilibrium are equivalent.
- Nash equilibria are thoroughly studied e.g. by Damme (1991).
- Since solution strategies are usually mixed strategies, and the space of mixed strategies is a convex subset of an Euclidean space, in this area one may find the most applications and connections with der wunderschöne Welt der Konvexität.

1.2 Strategic structure

Another field of research is concerned with the underlying structure of a game rather than with solution concepts. In this case one is interested with the notions *symmetry for a game* (Ham 2018 and citations therein) and of *equivalence between games* (Ham 2019 and citations therein).

The flavor is more abstract: one can for example define various notions of *games isomorphism* inducing on the space of games an equivalence relation making precise the idea of *games with the same strategic structure*, and look at the quotient space. Goforth and Robinson (2005) counted 144 classes of symmetric 2-player 2-strategy games up to *ordinal* isomorphisms.

2 Evolutionary Game Theory

Smith (1972) and **Smith and Price (1973)** apply game theory to animals behavior (they wonder why most intraspecies fights are non lethal), generalizing Normal Form Games to *Population Games*. The relevant solution concept is that of *Evolutionarily Stable Strategy* (ESS) (Smith 1974), stronger than that of Nash equilibrium.

- A very concise, very useful review is Sandholm 2017.
- Hofbauer and Sigmund (1998) provide a complete, classical reference; a modern comprehensive treatment is given by Sandholm (2010).
- Haigh (1975) publishes some results about the relation between ESSs and *stable* Nash equilibria, both for non-linear and linear payoffs[2].

[2] The definition of Nash equilibrium is purely game-theoretical, while that of ESS is intrinsically population-game-theoretical. The idea of ESS is very intuitive, and provides the *quid* necessary to make more solid (or stable, in a precise sense) the somewhat elusive notion of Nash equilibrium.

- A Normal Form Game defines a linear Population Game by *matching* (individuals of the population *meet* and play the game).
- Of interest can be Population Games known as *Playing the field*, which are not necessarily linear. Quoting Sandholm, *Maynard Smith observed that matching is a rather special sort of interaction in large populations. Instead, interactions in which each agent's payoffs are determined directly from all agents' behavior—what Maynard Smith terms "playing the field"—seem to be the rule rather than the exception. [...] some [examples], like congestion in highway networks, require payoffs to depend nonlinearly on the population state and so are mathematically inconsistent with a random matching approach. One might expect that moving from linear to nonlinear payoffs would lead to intractable models, but it does not. The dynamics studied here are nonlinear even when payoffs in the underlying game are not, so allowing nonlinear payoffs does not lead to a qualitative increase in the complexity of the analysis.*

3 Evolutionary Dynamics

So far both GT and EGT are concerned only with equilibria. **Taylor and Jonker (1978)** attempt to model the *dynamics* of a population game, that is the time evolution of the population state. They employ *imitation* as driving mechanism (successful strategies are likely to spread) and write a system of non-linear first-order differential equations, the *replicator* system. This is the first example of an *Evolutionary Dynamics.*

Again from Sandholm on the replicator system: *Schuster and Sigmund (1983), following Dawkins (1976), dubbed this model the replicator dynamic, and recognized the close links between this game-theoretic dynamic and dynamics studied much earlier in population ecology (Lotka 1920; Volterra 1931) and population genetics (Fisher 1930).*

This detour into mathematical biology and population dynamics greatly broadens the horizons of classical game theory; the ball is then back to the economists, who realize the evolutionary approach provides further conceptual foundation to the notion of Nash equilibrium, and allows to select among coexisting equilibria via global and local stability analysis. From this moment on biological and socio-economical interpretations interwine and enrich each other.

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