Parabolic fixed points: The Leau-Fatou Flower

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We want to study the dynamics of a holomorphic map f from the Riemann sphere to itself in a small neighborhood of a parabolic fixed point. The exposition follows closely [6, Chapter 10]. Other useful resources are [1, Chapter 6] and [2, Chapter 2].

The picture is the following: If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a globally defined holomorphic function and $\hat{z}$ is a fixed point with multiplier equal to one, then around $\hat{z}$ there exist non-unique "local basins of attraction", called *attracting petals*, each determining *uniquely* a global basin of attraction. Orbits in each basin converge to the fixed point along a direction determined by the corresponding petal. Similarly there exist *repelling petals*, defined as attracting petals for the locally well-defined and holomorphic inverse of f. Petals can be chosen so that their union is an open neighborhood of the fixed point, each petal intersecting precisely its neighbors, producing a "flower" pattern as in Fig. 3.

The plan is to build open regions having the fixed point on their boundary where the dynamics under f can be conjugated to (almost) a translation. The limit behavior of this conjugated dynamics is readily understood and mapped back in the original space.

Express f in a local chart, which can be chosen so that the fixed point corresponds to $z = 0$, and therein expand f in its converging power series

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots \tag{1}$$

Recall that the coefficient $\lambda = f'(0)$ is called *multiplier* of the fixed point, and that a fixed point is called *parabolic* if λ is a q-th root of unity, and $f^{\circ q} \neq \mathrm{id}$.

Definition 1. Let $f \in \mathrm{Hol}(\hat{\mathbb{C}})$ fix $\hat{z}$. The *multiplicity* of the fixed point $\hat{z}$ is the order of the first non-vanishing term of the expansion of $f(z) - z$ around $\hat{z}$.

Remark 2. Since the expansion of $f(z) - z$ around a fixed point in the above local chart reads

$$f(z) - z = (\lambda - 1)\, z + a_2 z^2 + a_3 z^3 + \cdots$$

the multiplicity of a fixed point of a non-constant holomorphic function is greater or equal than 2 if and only if the multiplier λ is equal to 1.

In the following we consider parabolic fixed points with multiplier equal to one or, equivalently, multiplicity greater or equal than 2. The multiplicity of the fixed point is denoted by

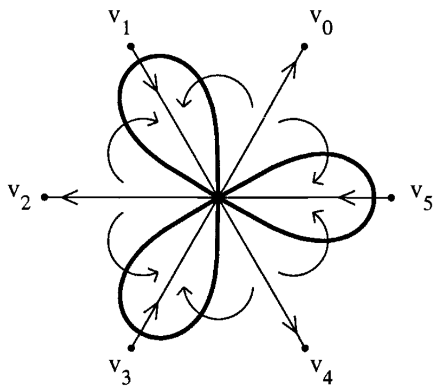

Figure 18. Schematic picture of a parabolic point of multiplicity $n+1=4$. (Here $a=-1$.) Each arrow indicates roughly how points are moved by f. The three attraction vectors are indicated by arrows pointing towards the origin, and the three repulsion vectors by arrows pointing away from the origin.

Figure 1: From [6, p. 105].

$n+1$, so that n is an integer greater or equal than 1; and the nonzero coefficient of the term of order $(n+1)$ in the expansion of f is denote by a. Thus in a neighborhood of a parabolic fixed point with multiplier equal to one we have

$$f(z) = z + az^{n+1} + \text{higher order terms} \tag{2}$$

with integer $n+1 \geq 2$ and complex $a \neq 0$.

Next we introduce an array of $2n$ vectors that will turn out to provide the possible directions and rate of convergence for orbits converging to the origin without actually reaching it.

Definition 3. Consider f as in (2). The *repulsion vectors* for f at the origin are the n-th roots of $+1/an$, and the *attraction vectors* for f at the origin are the n-th roots of $-1/an$.

Clearly there are n equally spaced attraction vectors, separated by n equally spaced repulsion vectors (Fig. 1); in particular the angle between a vector of a type and one of its neighbors of the opposite type is π/n. If v_0 is any repelling vector we can label the attraction and repulsion vectors as

$$v_j = v_0 e^{\pi i j/n}, \quad j = 0, \ldots, n-1, \quad \text{so that } v_j^n = (-1)^j/an \tag{3}$$

Thus v_j is an attraction vector (or simply *attracting*) if j is odd, and repelling if j is even.

Remark 4. By the inverse function theorem, the inverse f^{-1} of f is well defined and holomorphic in a neighborhood of the origin. By chain rule $\lambda_{f^{-1}} = 1/\lambda_f$, so f and f^{-1} have the same set of fixed points with multiplier equal to one. It is not hard to check that in a small enough neighborhood of one of them where f is invertible and expanded as in (2), f^{-1} is expanded as

$$f^{-1}(z) = z - az^{n+1} + \text{higher order terms} \tag{4}$$

Thus the attraction vectors for f at the origin are precisely the repulsion vectors for f^{-1} at the origin, and vice versa.

Let's make precise the idea that attraction (resp. repulsion) vectors provide the directions and rate of convergence for orbits converging under f (resp. f^{-1}) to the origin without actually reaching it; for this we need a definition and a lemma.

Definition 5. An orbit $\mathcal{O}_f(z_0) = \{z_0 \overset{f}{\mapsto} z_1 \mapsto \dots\}$ is said to converge to 0

- *non-trivially* if $z_k \to 0$ as $k \to \infty$, but $z_k \neq 0$ for all $k \geq 0$;
- *along an attraction vector* if $\lim_{k\to\infty} \sqrt[n]{k}\, z_k = v_j$ where v_j is an attraction vector for f. In this case we also write $z_k \sim v_j / \sqrt[n]{k}$.

Lemma 6. *An orbit $\mathcal{O}_f(z_0)$ converges to* 0 *non-trivially if and only if it converges to* 0 *along one of the attraction vectors.*

Proof. The implication [$\Leftarrow$] is obvious. Consider an attraction vector v_j, which is nonzero per definition, so that $\lim_{k\to\infty} \sqrt[n]{k}\, z_k = v_j \neq 0$. Assume $\mathcal{O}_f(z_0)$ converges trivially to the origin; then there exists $K \geq 0$ such that $z_k = 0$ for all $k \geq K$, so $\lim_{k\to\infty} \sqrt[n]{k} z_k = 0$, which is a contradiction.

To prove the other direction we need to build open regions having the fixed point on their boundary, each containing precisely one attraction vector, such that in each of those regions the dynamics under f can be conjugated to (almost) a translation in the direction of the conjugated corresponding attraction vector. The limit behavior of non-trivially converging orbits is readily understood in the conjugated space, and yields the desired result when mapped back in the original space. A similar technique is employed to prove Theorem 10, so we prove these results together later on. □

Definition 7. The *parabolic basin of attraction* for a fixed point $\hat{z}$ of $f \in \mathrm{Hol}(\hat{\mathbb{C}})$ with multiplier equal to one and for one of its attraction vectors v_j is

$$\mathcal{A}(\hat{z}, v_j) = \{z \in \hat{\mathbb{C}} \text{ s.t. } \mathcal{O}_f(z) \text{ converges to } \hat{z} \text{ along } v_j\} \tag{5}$$

When there is no risk of ambiguity about the considered fixed point we simply write $\mathcal{A}_j$ for $\mathcal{A}(\hat{z}, v_j)$. The *immediate basin* is defined to be the unique connected component of $\mathcal{A}_j$ which maps into itself under f. See Fig. 2.

Lemma 8. *Parabolic basins of attractions*

1. *are fully invariant;*
2. *are disjoint;*
3. *are open;*
4. *are contained in Fatou(f);*
5. *have boundary contained in Julia(f).*

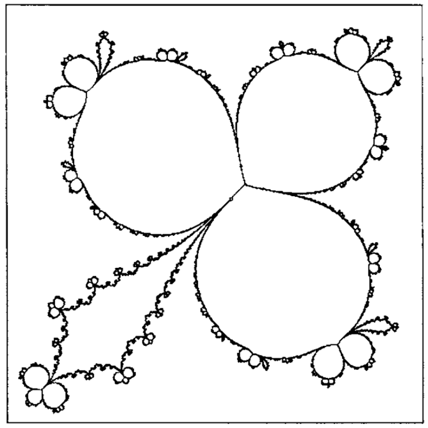

Figure 19. Julia set for $f(z) = z^5 + (.8+.8i)z^4 + z$. This map has a parabolic fixed point of rotation number zero and petal number three at $z = 0$ (and also an attracting fixed point at $z = -.8-.8i$). The immediate basins for the three attraction vectors resemble balloons, pulled together at the parabolic point and separated by the three repulsion vectors.

Figure 2: From [6, p. 106].

Proof.

1. We have to show that $z_0 \in \mathcal{A}_j \Leftrightarrow f(z_0) \in \mathcal{A}_j$. Let $z_0 \in \mathcal{A}_j$ and denote $z_k = f^{\circ k}(z_0)$. Then $\lim_{k\to\infty} \sqrt[n]{k}\, z_k = v_j$. Let $w_k = z_{k+1}$; it follows that $\lim_{k\to\infty} \sqrt[n]{k}\, w_k = \lim_{k+1\to\infty} \sqrt[n]{k+1} \frac{\sqrt[n]{k}}{\sqrt[n]{k+1}} z_{k+1} = v_j$, i.e. $w_0 \equiv z_1 \equiv f(z_0) \in \mathcal{A}_j$. The converse is similar.

2. Let $z \in \mathcal{A}_j \cap \mathcal{A}_h$, then by uniqueness of the limit $\lim_{k\to\infty} \sqrt[n]{k}\, z_k = v_j = v_h$, so $j \equiv h (\mathrm{mod}\ 2n)$ and $\mathcal{A}_j = \mathcal{A}_h$.

3. We could not find an immediate way to show that parabolic basins are open. For the moment we accept this result, and prove it later on (Corollary 15), when the Parabolic Flower Theorem is established.

4. Let $z_0 \in \mathcal{A}_j$, which is open. Take a neighborhood U of z_0 contained in $\mathcal{A}_j$: therein $z_k \sim v_j / \sqrt[n]{k}$, i.e. the sequence of iterates $f^{\circ k}|_U$ converges uniformly to the constant map 0, so $z_0 \in \mathrm{Fatou}(f)$.

5. Consider two cases for a point on a basin boundary. First let $z_0 \in \partial\mathcal{A}_j$ such that its orbit converges to 0. Since z_0 is on a basin boundary, and basins are open and disjoint, z_0 is not in any other basin, so its orbit converges to 0 *trivially*, i.e. it eventually lands exactly on the parabolic fixed point, which belongs to the Julia set [6, Lemma 4.7]. Since the Julia set is fully invariant [6, Lemma 4.3], the whole orbit is contained in the Julia set.

   Secondly consider a point z_0 on a basin boundary whose orbit does not converge to the origin; then we can extract a subsequence $z_{k(i)}$ which is bounded away from the

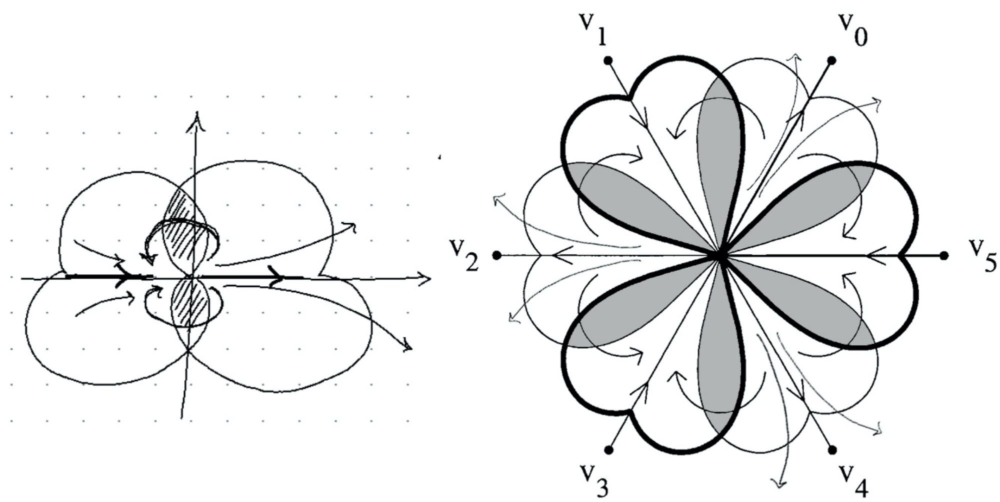

Figure 3: Flowers with $2n = 2$ (left) and $2n = 6$ (right) petals. If $n = 1$, the intersection of the two petals has two simply connected components. If $n \geq 2$, each petal intersects each of its neighbors in one simply connected region, and is disjoint from the other petals. Each arrow indicates roughly how points are moved by f. Adapted from [6, p. 112].

origin. The sequence of iterates does converge to 0 throughout $\mathcal{A}_j$, hence the family $\{f^{\circ k}\}$ cannot be normal in any neighborhood of z_0, which is then a Julia point.

This concludes the proof of Lemma 8. □

It is often convenient to have a purely local analog for the global concept of "basin of attraction".

Definition 9. Let $f \in \mathrm{Hol}(\hat{\mathbb{C}})$ have a fixed point $\hat{z}$ with multiplicity $n+1 \geq 2$ and let v be an attraction vector for f at $\hat{z}$. Let N be an open neighborhood of $\hat{z}$ where f^{-1} is well-defined and holomorphic. An open set $P \subset N$ is called an *attracting petal* for f and v at $\hat{z}$ if

- $f(P) \subseteq P$, and
- an orbit $\mathcal{O}_f(z_0)$ eventually enters P if and only if $\mathcal{O}_f(z_0)$ converges to $\hat{z}$ along v.

Similarly, an open set $Q \subset f(N)$ is called a *repelling petal* for f and for the repulsion vector u if it is an attracting petal for $f^{-1} : f(N) \to N$ and for u.

The following result was proved in a preliminary form by Leopold Leau [1897], and in increasingly satisfactory forms by Julia [1918] and Fatou [1919-1920].

Theorem 10 (Parabolic Flower)**.** *Within any neighborhood of a fixed point $\hat{z}$ with multiplicity $n+1 \geq 2$ there exist $2n$ simply connected petals $\mathcal{P}_j$, $j = 0, \ldots, 2n-1$, attracting or repelling according to whether j is even or odd. The petals can be chosen so that $\bigcup_j \mathcal{P}_j \cup \{\hat{z}\}$ is an open neighborhood of $\hat{z}$. If $n = 1$, the intersection of the two petals has two simply connected components. If $n \geq 2$, each petal intersects each of its neighbors in one simply connected region, and is disjoint from the other petals.*

Remark 11. Let's have a look at the global picture before jumping into the proof. If $f \in \mathrm{Hol}(\hat{\mathbb{C}})$ is a globally defined holomorphic function and $\hat{z}$ is a fixed point of multiplicity $n+1 \geq 2$, then each attracting petal $\mathcal{P}_j$ about $\hat{z}$ determines a corresponding parabolic basin of attraction $\mathcal{A}_j$, consisting of all z_0 for which the orbit $\mathcal{O}_f(z_0)$ eventually lands in $\mathcal{P}_j$, and hence converges to the fixed point from the associated direction v_j.

Remark 12. Orbits in an attracting petal stay therein and converge to 0 along the corresponding attraction vector. Orbits in a repelling petal converge to 0 under f^{-1} (which is locally well defined and holomorphic) along the corresponding repulsion vector. Under f these orbits can either leave the repelling petal without entering the attracting petal; or enter the intersection between the repelling petal and a neighboring attracting petal, to eventually converge to 0 along an attracting direction. Thus around a parabolic fixed point with multiplier equal to one no spiral orbit nor periodic orbit is allowed - more precisely, there exists a neighborhood of the fixed point that does not contain any periodic orbit (but for the trivial one, namely the fixed point itself). This intuitively explains the arrows in Figures 1 and 3, and is proved rigorously in due course, see Corollary 14.

Remark 13. In the following proof we first build regions, denoted by $P_j(R)$, that are petals according to Definition 9 but do not fulfill the intersection property stated in Theorem 10; next we extend those regions to "fatter" regions $\mathcal{P}_j(R) \supset P_j(R)$, that are again petals and furthermore fulfill the desired intersection property.

Proof. (Lemma 6 and Parabolic Flower Theorem)
The idea is to conjugate the dynamics to almost translation in open regions having 0 on the boundary. Let

$$\phi(z) = w = \frac{c}{z^n}, \quad c = -\frac{1}{an} \tag{6}$$

Recall the definition of attracting and repelling vectors:

$$v_j^n = \frac{(-1)^j}{an} \tag{7}$$

with v_j attracting (resp. repelling) if j is odd (resp. even). Thus

$$\phi(v_j) = (-1)^{j+1} \tag{8}$$

The goal is to label the branches of of the multi-valued function

$$\phi^{-1}(w) = \sqrt[n]{\frac{c}{w}} \tag{9}$$

To do so let's cover the punctured plane with open sectors of angle $2\pi/n$ bounded by two consecutive vectors of the same type:

$$\Delta_j := \left\{ z \in \hat{\mathbb{C}} \text{ s.t. } z = re^{i\theta}v_j, \quad r > 0,\, |\theta| < \frac{\pi}{n} \right\} \tag{10}$$

The plan is now the following. Consider an *attracting* vector v_j:

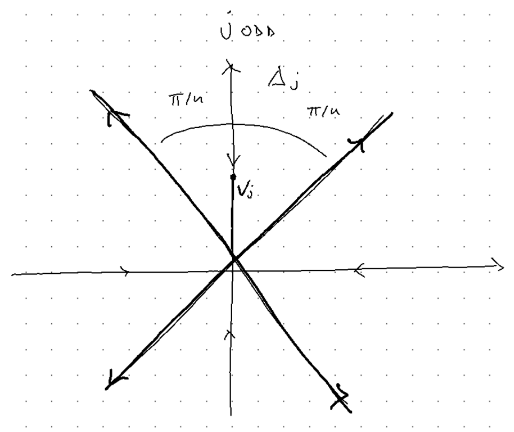

Figure 4: Definition of Δ_j

1. In Δ_j, ϕ is invertible;
2. Study the dynamics of $\phi \circ f \circ \phi^{-1}$, called *conjugated dynamics*, in $\phi(\Delta_j)$;
3. Find a "nice" region for the conjugated dynamics (invariant and where orbits have a simple asymptotic behavior);
4. Go back up via ϕ^{-1}.

1. In Δ_j, ϕ is invertible It is not hard to check that

$$\phi : \Delta_j \xrightarrow{\sim} \mathbb{C} \setminus \mathbb{R}_- \tag{11}$$

is a biholomorphism, where $\mathbb{R}_- = (-\infty, 0]$ (recall that we choose v_j attracting, so j is odd). Apply ϕ to a point $z \in \Delta_j$:

$$\phi\left(re^{i\theta}v_j\right) = \frac{c}{r^n e^{in\theta} v_j^n} = \frac{1}{r^n} e^{-i\theta n} \tag{12}$$

Hence $|\theta| < \pi/n$ implies $n\theta \neq \pi (\mathrm{mod} 2\pi)$, and the inverse of (11) is

$$\psi_j : \mathbb{C} \setminus \mathbb{R}_- \xrightarrow{\sim} \Delta_j, \quad Re^{i\alpha} \mapsto re^{i\theta}v_j \tag{13}$$

with $R > 0$, $\alpha \neq \pi (\mathrm{mod} 2\pi)$, $r = 1/\sqrt[n]{R}$ and $\theta = -\alpha/n$.

The ray $\{z \in \Delta_j : z = r\, v_j, r > 0\}$, i.e. the set of points in Δ_j with $\theta = 0$, is mapped via ϕ to the positive real axis, with $\lim_{r \to 0} \phi(r\, v_j) = \infty$ and $\lim_{r \to \infty} \phi(r\, v_j) = 0$. Similarly, ϕ maps the rays bounding Δ_j to the negative real axis. Hence ϕ "opens and inverts" Δ_j. It is also not hard to check that the "right" half of Δ_j, namely the points obtained rotating the direction of v_j by an angle θ fulfilling $-\pi/n < \theta < 0$, are mapped onto the upper half plane; and that the "left" half of Δ_j is mapped onto the lower half plane. See Fig. 5.

It is useful to investigate the image via ψ_j of a line with constant imaginary part in the w-plane, namely $S_j(y) := \psi_j\{w = x + iy : x \in \mathbb{R}, y = \text{const} \neq 0\}$. If $y > 0$, $S_j(y)$ must belong

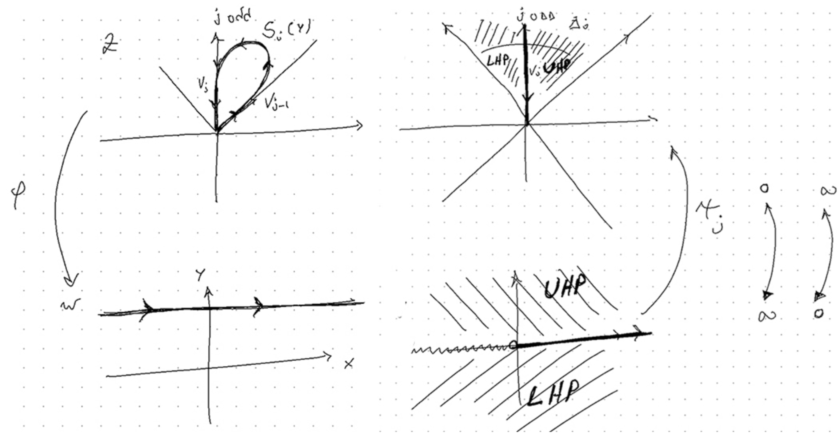

Figure 5: Left: $S_j(y) := \psi_j\{w = x + iy \,:\, x \in \mathbb{R}, y = \text{const}\}$. Right: $\phi : \Delta_j \xrightarrow{\sim} \mathbb{C} \setminus \mathbb{R}_-$

to the right half of Δ_j, and must converge to 0 when $x \to \pm\infty$. Furthermore the bigger y is, the closer should $S_j(y)$ be to the origin. Intuitively, $S_j(y)$ is tangent to the direction given by v_j for big values of x, and to the direction given by v_{j-1} for big values of $-x$; we make this precise in the following. See left column of Figs. 5-6.

Similarly (right column of Fig. 6), consider $Q_j(x) := \psi_j\{w = x + iy : x = \text{const} > 0, y \in \mathbb{R}\}$. The point of $Q_j(x)$ corresponding to $y = 0$ lies on the direction of v_j in z-space, the closer to the origin the bigger x is; and $Q_j(x)$ converges to zero as $|y| \to \infty$.

2. Study the conjugated dynamics in $\phi(\Delta_j)$ To study f close to the origin in Δ_j we need to look outside a large disk in the slit w-plane, see fig. (7). Let

$$F_j := \phi \circ f \circ \psi_j \tag{14}$$

Recall that $f(z) = z(1 + az^n + o(z^n))$ as $z \to 0$ and that $\psi_j(w) = \sqrt[n]{\frac{c}{w}}$ exists unique in Δ_j for $w \in \mathbb{C} \setminus \mathbb{R}_-$. So

$$f \circ \psi_j(w) = \sqrt[n]{\frac{c}{w}} \left(1 + \frac{ac}{w} + o\left(\frac{1}{w}\right)\right) \text{ as } |w| \to \infty$$

$$\begin{aligned} F_j(w) &= \phi\,(\text{the above line}) \\ &= c\frac{w}{c}\left(1 + \frac{ac}{w} + o\left(\frac{1}{w}\right)\right)^{-n} \\ &= w\left(1 + \frac{1}{w} + o\left(\frac{1}{w}\right)\right) \text{ as } |w| \to \infty \end{aligned}$$

where we used the expansion of $(1 + kx)^{-n}$ around $x = 0$, and the fact that $-nac = +1$. Thus

$$F_j(w) = w + 1 + o(1) \text{ as } |w| \to \infty \tag{15}$$

where $o(1)$ is a function of w that goes to 0 as $|w| \to \infty$[1].

[1]It can be shown, but we do not need it, that this remainder term is $O\left(1/\sqrt[n]{|w|}\right)$. See [6, p. 107]

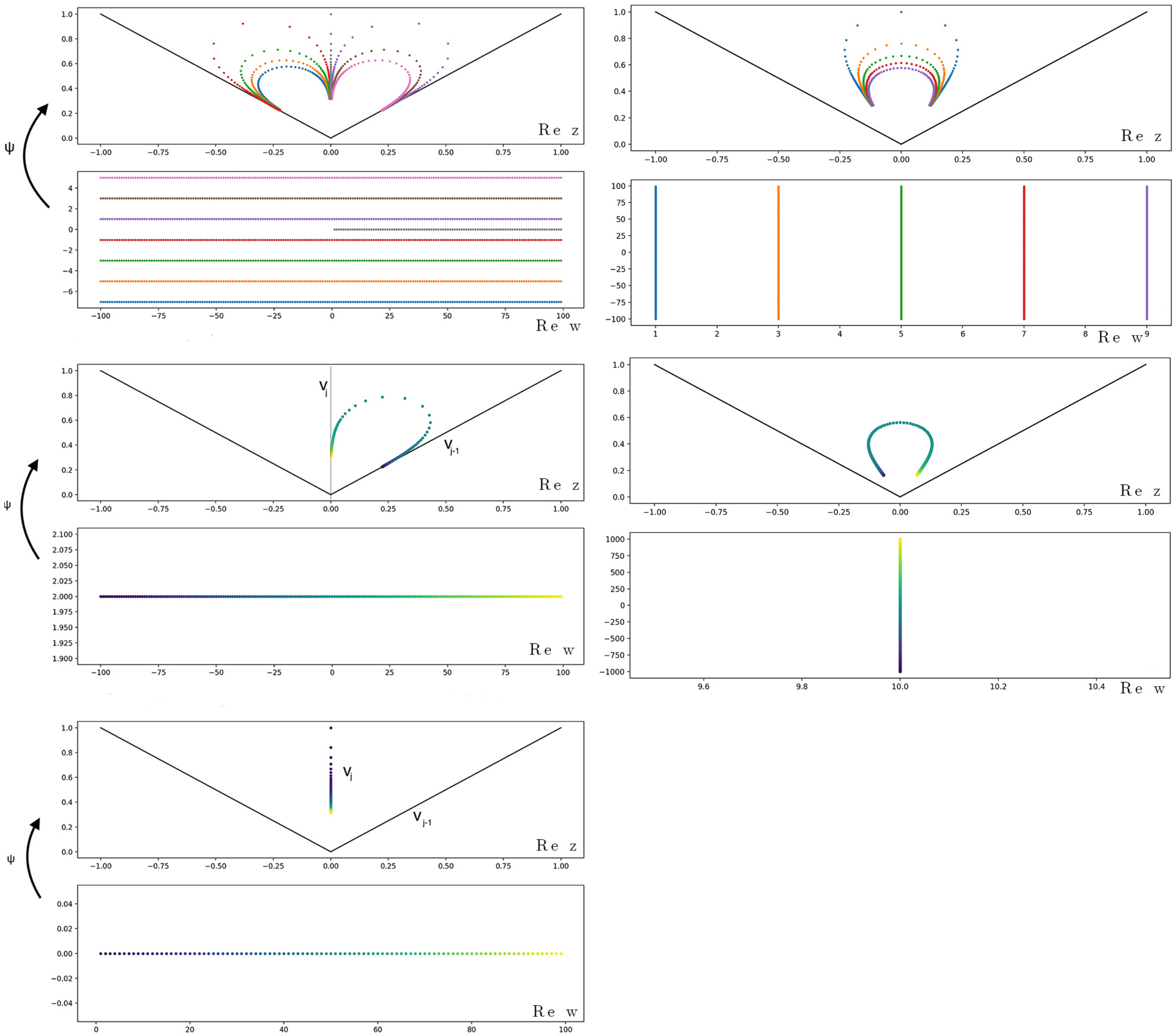

Figure 6: Images via $\psi_j : \mathbb{C}\setminus\mathbb{R}_- \xrightarrow{\sim} \Delta_j, w \mapsto z$ of discrete sets of points with the same real or imaginary part in the slit w-plane. Top left: Bigger $|\mathrm{Im} w|$ in w-space give tighter curves in z-space. Middle left: The curve in z-space is tangent to v_{j-1} for big negative values of Re w in w-space (dark colors), and tangent to v_j for big positive values of Re w (lighter colors), for Im $w > 0$. Bottom left: ψ_j maps the positive real numbers to the direction determined by v_j (j odd; in this case $v_j \propto i = (0, 1)$, and its direction is the imaginary axis in z-space), swapping ∞ and 0. Top right: as $\mathrm{Re}\, w > R > 0$ increases in w-space, the image curves in z-space get tighter. Bottom right: for fixed $\mathrm{Re}\, w > 0$, $\psi_j(w) \to 0$ as $|\mathrm{Im}\, w| \to \infty$.

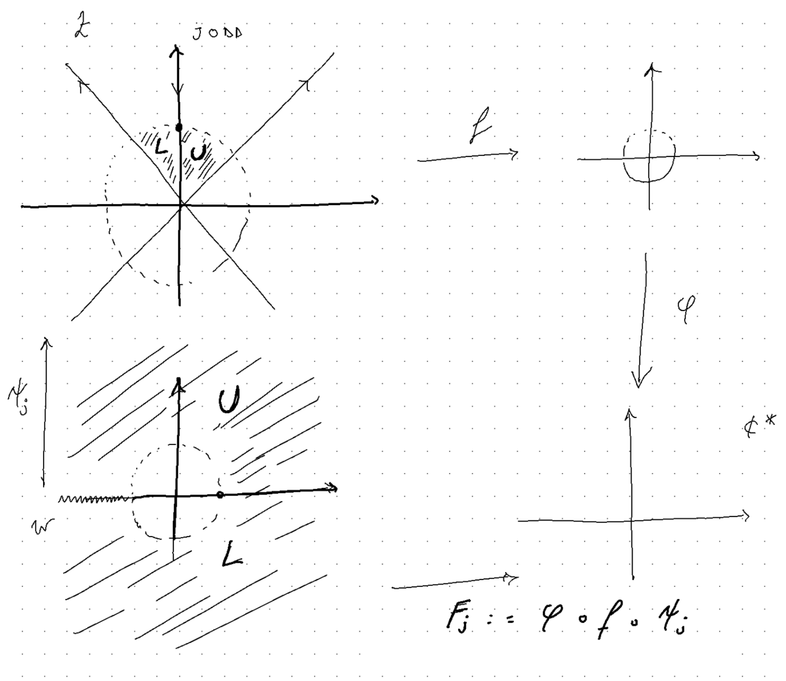

Figure 7: Define F_j away enough from the origin in the slit plane so that ψ_j lands in a given neighborhood of the origin.

Hence, the conjugated dynamics for $|w| \to \infty$ is “almost” a translation by $+1$ in the real direction, that is the direction of $\phi(v_j)$. In particular there exists $R > 0$ such that

$$|F_j(w) - w - 1| < \frac{1}{2} \text{ if } |w| > R \tag{16}$$

Recalling that $\mathrm{Re}(z) \leq |z|$ we have

$$\mathrm{Re}(w + 1 - F_j(w)) \leq |w + 1 - F_j(w)| = |F_j(w) - w - 1| < \frac{1}{2} \text{ if } |w| > R$$

so

$$\mathrm{Re}\, F_j(w) > \mathrm{Re}\, w + \frac{1}{2} \quad \text{when } |w| > R \tag{17}$$

And translating this back in the z plane via $F_j(w) = \phi \circ f(z)$ with $w = \phi(z)$ we have

$$\mathrm{Re}\, \phi\,(f(z)) > \mathrm{Re}\, \phi(z) + \frac{1}{2} \quad \text{when } |z| \text{ is small enough} \tag{18}$$

Corollary 14 (No periodic orbits)**.** *A parabolic fixed point with multiplier equal to one has a neighborhood that does not contain any periodic orbit but for the trivial one.*

Proof. Let U be a neighborhood of the origin small enough for (18) to hold. If there was a

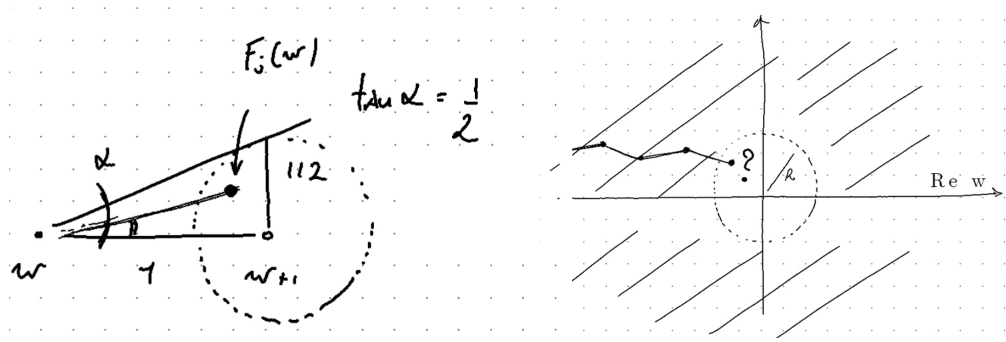

Figure 8: Left: The slope of the vector $F_j(w) - w$ is bounded. Right: It is not enough to require $|w| > R$ for the constraint (17) to hold true along the F_j-orbit of a point for which it holds true.

k-periodic orbit $z_0 = f^{\circ k}(z_0)$ *completely contained in* U with $k \geq 1$ and $z_0 \neq 0$ then

$$\begin{aligned} \operatorname{Re}\phi(z_1) &> \operatorname{Re}\phi(z_0) + \frac{1}{2} \\ \operatorname{Re}\phi(z_2) &> \operatorname{Re}\phi(z_1) + \frac{1}{2} > \operatorname{Re}\phi(z_0) + 2 \cdot \frac{1}{2} \\ \operatorname{Re}\phi(z_0) = \operatorname{Re}\phi(z_k) &> \operatorname{Re}\phi(z_0) + \frac{k}{2} \end{aligned} \tag{19}$$

which is absurd. □

Thus as a consequence of (15) (the conjugated dynamics is almost a translation in the positive real direction) we obtained (17) (the real part increases along the conjugated orbit), where both statements hold for $|w|$ big enough, i.e. $|z|$ small enough. We can derive a similar constraint on the slope of the vector $F_j(w) - w$, saying that the absolute value of the variation of the imaginary party is smaller than the increase of the real part along the conjugated orbit, for $|w|$ big enough. Eq. (16) says that $F_j(w) \in B_{w+1}(\frac{1}{2})$, i.e. that the image of w under F_j lands in the ball centered at $w+1$ of radius $1/2$, if $|w| > R$, so that $|\text{slope}| < \frac{1}{2}$. See fig. (8). Now

$$\text{slope} = \frac{\operatorname{Im} F_j(w) - \operatorname{Im} w}{\operatorname{Re} F_j(w) - \operatorname{Re} w}$$

the denominator being positive, so (this coarse estimation is enough)

$$|\operatorname{Im} F_j(w) - \operatorname{Im} w| < \operatorname{Re} F_j(w) - \operatorname{Re} w \quad \text{ when } |w| > R \tag{20}$$

We are now in the position to show that an orbit converging to zero non trivially converges along an attraction vector, and to prove the existence of petals.

3. Find a "nice" region for the conjugated dynamics We look for a region in the slit w-plane in which, if the constraint (17) holds true for a point, it keeps holding true for

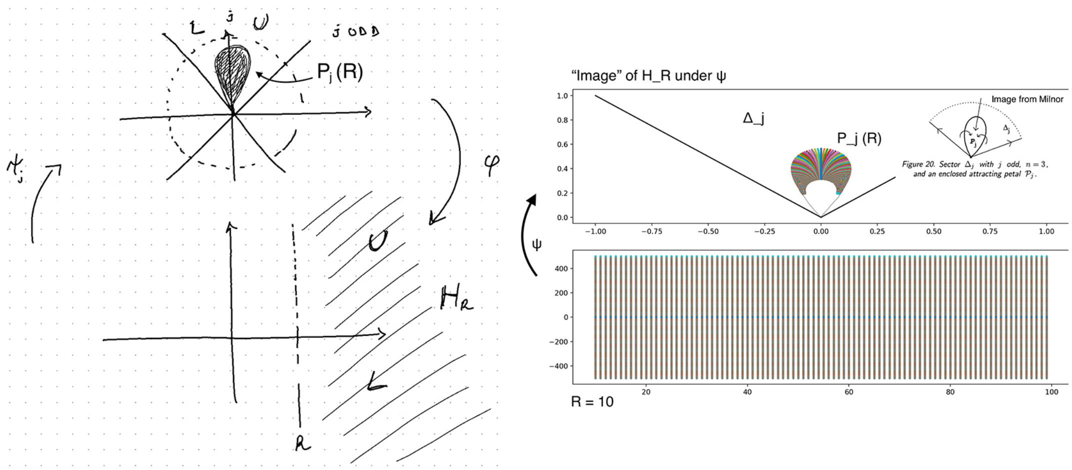

Figure 9: $P_j(R) = \psi_j(H_R)$

the orbit of this point under F_j. Observe (Fig. 8, right) that it is not enough to require $|w| > R$. In the following, making use of Eq. (20), we will consider a fatter region, but for the moment we play it safe and consider

$$H_R = \{w \in \mathbb{C} : \operatorname{Re} w > R > 0\} \tag{21}$$

Clearly $F_j(H_R) \subset H_R$; for if $w_0 \in H_R$ then $|w_0| > R$, so eq. (17) holds and $F_j(w_0) \in H_R$. Consider the image of this region under ψ_j:

$$P_j(R) := \psi_j(H_R) = \{z \in \Delta_j : \operatorname{Re} \phi(z) > R\} \tag{22}$$

By restricting the domain of ψ_j to points w with bigger real part we are squeezing the image into a smaller region of Δ_j, see Fig. 6 top right and Fig. 9.

Claim The regions $P_j(R)$ (with j odd) are attracting petals for f and for the attraction vector v_j at the origin, according to Definition 9.

Just like H_R is invariant under the F_j-dynamics, $P_j(R)$ is invariant under the f-dynamics: if $z \in P_j(R)$ then (18) holds and $f(z) \in P_j(R)$. ($\star$)

Furthermore, as in the proof of Corollary 14, if $z_0 \in P_j(R)$ then $z_k \in P_j(R) \subset \Delta_j$ for all $k \geq 0$ and

$$\operatorname{Re} \phi(z_k) > \operatorname{Re} \phi(z_0) + \frac{k}{2} \tag{23}$$

so $f^{\circ k}(z_0) = z_k = \psi_j \circ \phi\,(z_k) \to \psi_j(\infty) = 0$ as $k \to \infty$, i.e. the successive iterates of f restricted to $P_j(R)$ converge to the constant map $P_j(R) \to 0$. Since $P_j(R)$ is invariant under f and $0 \notin P_j(R)$, this means that an orbit that enters $P_j(R)$ converges to zero non-trivially. ($\star\star$)

Conversely consider an orbit $\mathcal{O}_f(z_0)$ converging to zero non trivially, where z_0 is a generic point in $\hat{\mathbb{C}}$; by Eq. (18) then $\operatorname{Re} \phi(z_{k+1}) > \operatorname{Re} \phi(z_k) + \frac{1}{2}$ whenever k is large enough. In particular there exists an m such that $\operatorname{Re}\phi(z_m) > R$, hence $z_m \in P_j(R) \subset \Delta_j$ for some odd j. Since $f(P_j(R)) \subset P_j(R)$, it follows that z_k belongs to this same $P_j(R)$ for all $k \geq m$. In particular this means that an orbit converging to the origin along v_j (and thus non-trivially) eventually enters $P_j(R)$. ($\star\star\star$)

Let $\{z_k\}$ converge to zero non-trivially and consider the sequence $w_k = \phi(z_k)$; then from the discussion above $w_k \in H_R$ and $w_{k+1} = \phi(z_{k+1}) = \phi \circ f(z_k) = \phi \circ f \circ \psi_j(w_k) = F_j(w_k)$ for $k \geq m$. By Eq. (17) $\operatorname{Re} w_k \to \infty$ and in particular $|w_k| \to \infty$ as $k \to \infty$, so by Eq. (15) we have that $w_{k+1} - w_k \xrightarrow[k\to\infty]{} 1$, and

$$1 = \lim_{k\to\infty} \frac{1}{k} \sum_{h=0}^{k-1} w_{h+1} - w_h = \lim_{k\to\infty} \frac{w_k - w_0}{k} = \lim_{k\to\infty} \frac{w_k}{k}$$

In other words, the asymptotic behavior for a conjugated non-trivially converging orbit is

$$\lim_{k\to\infty} \frac{w_k}{k} = 1, \quad \text{or } w_k \sim k \text{ as } k \to \infty \tag{24}$$

4. Go back up via ϕ^{-1} Finally we can map this result back in z-space: consider again an orbit $\{z_k\}$ converging to zero non-trivially, so

$$\begin{aligned} w_k = \phi(z_k) = \frac{c}{z_k^n} &\Rightarrow z_k^n \sim \frac{-1}{na}\frac{1}{w_k} \sim \frac{-1}{na}\frac{1}{k} \\ &\Rightarrow z_k \sim \sqrt[n]{\frac{-1}{na}}\frac{1}{\sqrt[n]{k}} \end{aligned}$$

For k big enough z_k is inside some $P_j(R)$ (with j odd), so the first square root of the right hand side of the last line is uniquely defined and equal to one of the attraction vectors, proving Lemma 6. Furthermore, considering ($\star$), ($\star\star$) and ($\star\star\star$) in the discussion above, this also proves that $P_j(R)$ is an attracting petal.

Since f and f^{-1} have the same set of fixed points with multiplier equal to one, and the same array of attraction-repulsion vectors with swapped roles, this also proves that an orbit $z_0 \overset{f^{-1}}{\mapsto} z_1 \mapsto \cdots$ under the inverse f^{-1} converges to zero non-trivially if and only if it converges to zero along one of the *repulsion* vectors of f (which is indeed an attraction vector for f^{-1}); and that a repelling petal exists in the intersection of a small neighborhood of the origin with Δ_j, with j even.

Work out the details as an exercise. Start by showing that $\phi : \Delta_j \xrightarrow{\sim} \mathbb{C} \setminus \mathbb{R}_+$ is a biholomorphism if j is even (so the v_j defining Δ_j is repelling). The direction of v_j is mapped to the negative real axis. Recall from Eq. (4) that the expansion of f^{-1} differs up to order $n+1$ from that of f by the sign of a and show that the conjugated dynamics results in $w \mapsto w - 1 + o(1)$ as $|w| \to \infty$. Derive the analogues of constraints (17)-(20) and use them to discuss the invariance of $-H(R) := \{w \in \mathbb{C} : \operatorname{Re} w < R < 0\}$ under the conjugated dynamics, and of its image $\psi_j(-H(R))$ under the f^{-1}-dynamics. Finally show that $\lim_{k\to\infty} w_k/k = -1$ for a non-trivially converging orbit, and translate this result in the z-space.

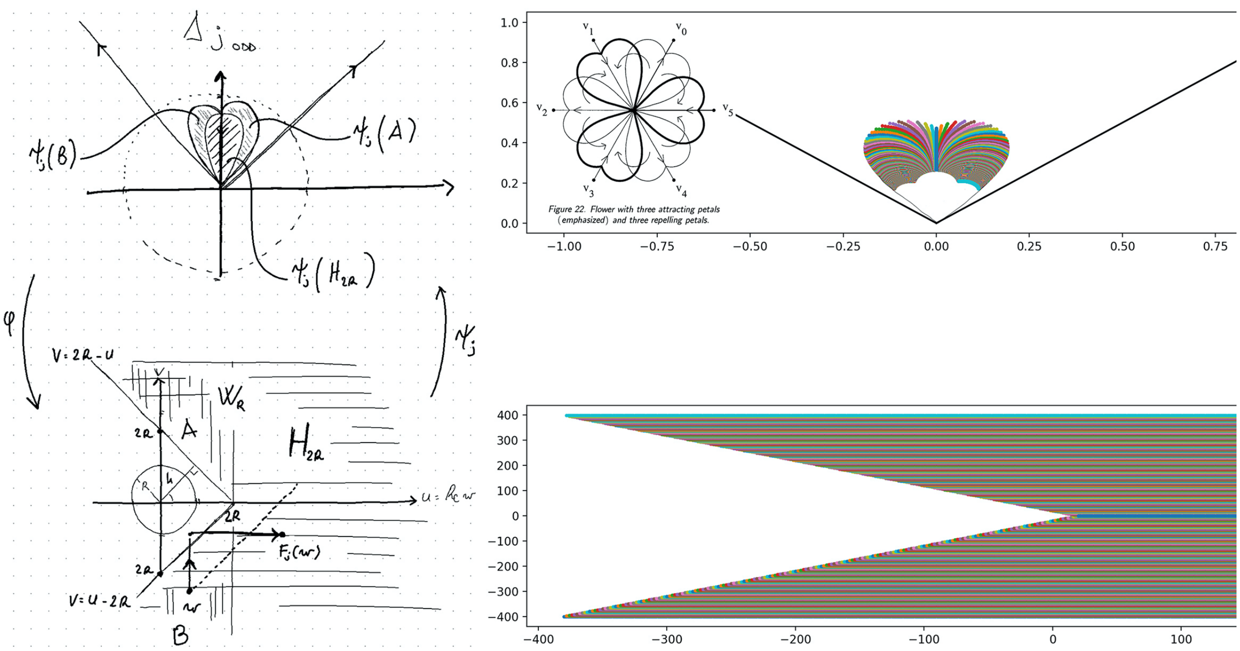

Figure 10: $\mathcal{P}_j(R) = \psi_j(W_R)$

To conclude the proof of the Parabolic Flower theorem we need to consider *fatter* petals, preserving the dynamical behavior we just described and in addition intersecating as in the statement of the theorem. The idea is to use Eq. (20) to extend the region H_R yet preserving the invariance under the conjugated dynamics. Define

$$W_R := \{w = u + iv \in \mathbb{C} : u + |v| > 2R\} \supset H_{2R} \tag{25}$$

By a simple geometrical argument (see Fig. 10)

$$h := \inf_{w \in W_R} |w| = 2R \sin \pi/4 = \sqrt{2}R > R \tag{26}$$

This implies that $F_j(W_R) \subset W_R$ for each odd j: Eq. (20) holds true for $w \in W_R$, so the absolute shift in the imaginary part is smaller than the positive shift in the real part, and $F_j(w)$ lies inside W_R. See again Fig. 10.

It is now not hard to see that $\mathcal{P}_j(R) := \psi_j(W_R)$ is an attracting petal for each odd j: invariance follows from the invariance of W_R under the conjugated dynamics; and the second property defining a petal follows from the fact that $\mathcal{P}_j(R) \supset P_j(2R)$, the latter known to be a petal.

First, let $z \in \mathcal{P}_j(R)$; then $\phi(z) \in W_R$ and $W_R \ni F_j(\phi(z)) = \phi \circ f \circ \psi_j \circ \phi(z) = \phi(f(z))$, so $\psi_j(\phi(f(z))) = f(z) \in \psi_j(W_R) = \mathcal{P}_j(R)$ i.e. $f(\mathcal{P}_j(R)) \subset \mathcal{P}_j(R)$. Secondly, note that an orbit that enters $\mathcal{P}_j(R)$ eventually enters $P_j(2R)$, thus converging to zero along v_j. Finally, any orbit converging to zero along v_j eventually enters $P_j(2R)$, hence it also enters $\mathcal{P}_j(R)$.

Similarly, working with f^{-1} and j even leads to the repelling petals $\psi_j(-W_R)$, with $-W_R := \{w : -w \in W_R\} = \{w = u + iv \in \mathbb{C} : -u + |v| > 2R\}$.

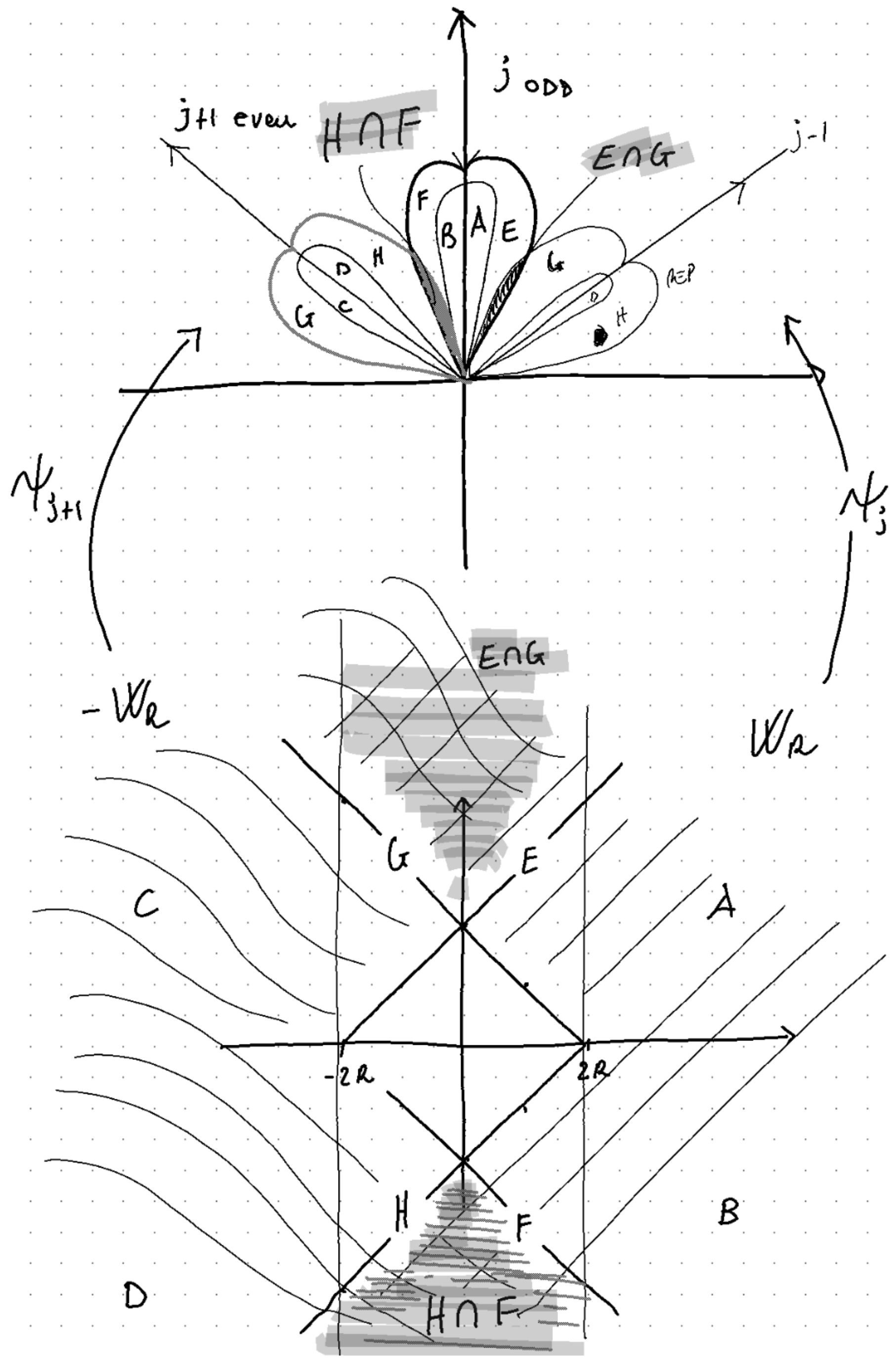

Figure 11: The intersection $W_R \cap (-W_R)$ is a disjoint union $V_R^+ \sqcup V_R^-$, where V_R^+ is the simply-connected open V-shaped region consisting of all $u + iv$ in the upper-half plane with $v > |u| + 2R$ (denoted here by $E \cap G$), and V_R^- is the reflection in the lower-half plane (denoted here by $H \cap F$).

The intersection $W_R \cap (-W_R)$ is a disjoint union $V_R^+ \sqcup V_R^-$, where V_R^+ is the simply-connected open V-shaped region consisting of all $u + iv$ in the upper-half plane with $v > |u| + 2R$ (denoted by $E \cap G$ in Fig. 11), and V_R^- is the reflection in the lower-half plane (denoted by $H \cap F$ in Fig. 11). If j is odd (v_j attracting), the right half of a petal corresponds to points w with $\operatorname{Im} w > 0$, and the left half of a petal corresponds to points w with $\operatorname{Im} w > 0$; and vice versa for j even. This is summarized in the following table:

	j odd	j even
Right	upper	lower
Left	lower	upper

Thus, for j odd and regions F, H as in Fig. 11,

$$\mathcal{P}_j(R) \cap \mathcal{P}_{j+1}(R) = \psi_j(W_R) \cap \psi_{j+1}(-W_R) = \psi_j(F) \cap \psi_{j+1}(H) = \\ \psi_j(H \cap F) = \psi_{j+1}(H \cap F) = \psi_j(V_R^-) = \psi_{j+1}(V_R^-) \quad (27)$$

Similarly

$$\mathcal{P}_j(R) \cap \mathcal{P}_{j-1}(R) = \psi_j(W_R) \cap \psi_{j-1}(-W_R) = \psi_j(E) \cap \psi_{j-1}(G) = \\ \psi_j(E \cap G) = \psi_{j-1}(E \cap G) = \psi_j(V_R^+) = \psi_{j-1}(V_R^+) \quad (28)$$

concluding the proof of the theorem. □

Corollary 15. *Parabolic basins of attraction are open.*

Proof. The Parabolic Flower Theorem grants the existence of a (non-unique) petal P_j associated to a parabolic fixed point with multiplier equal to one and to any of its attraction vectors v_j. Recall that a generic orbit is eventually absorbed by P_j if and only if it converges to the fixed point along v_j. The corresponding parabolic basin of attraction $\mathcal{A}_j$ is the set of points whose orbit converges to the fixed point along v_j. Thus

$$\begin{aligned} \mathcal{O}_f(z) \text{ eventually enters } P_j &\iff \mathcal{O}_f(z) \text{ converges to the fixed point along } v_j \\ &\iff z \in \mathcal{A}_j \end{aligned} \quad (29)$$

The set of points z whose orbit $\mathcal{O}_f(z)$ eventually enters P_j is the union of the preimages of P_j via all the k-fold iterates of f, so

$$\mathcal{A}_j = \bigcup_{k=0}^{\infty} \left(f^{\circ k}\right)^{-1}(P_j) \quad (30)$$

which is open since P_j is open and each $f^{\circ k}$ is holomorphic, in particular continuous. □

To conclude we can say something for the case of a parabolic fixed point (as usual, without lost of generality, working in a chart where it corresponds to the origin) whose multiplier is a primitive q-th root of unity, with $q > 1$. Everything we said so far applies to the non-identity q-fold iterate $f^{\circ q}$, for the multiplier λ_q of the fixed point with respect to $f^{\circ q}$ is 1:

$$\lambda_q = (f^{\circ q})'(0) = (f'(0))^q = \lambda^q = 1$$

Lemma 16. *If the multiplier of a parabolic fixed point $\hat{z}$ is a primitive q-th root of unity, then the number n of attraction vectors for $f^{\circ q}$ at $\hat{z}$ is a multiple of q; equivalently, the multiplicity $n+1$ of $\hat{z}$ as a fixed point of $f^{\circ q}$ is congruent to 1 modulo q.*

Proof. Expand f and $f^{\circ q}$ around the fixed point up to order $n+1$, which is per definition the order of the first non-vanishing term of the expansion of $f^{\circ q}(z) - z$. Thus $f^{\circ q}(z) = z + az^{n+1} + \ldots$ and $f(z) = \lambda z + \sum_{m=2}^{n+1} b_m z^m + \ldots$. Up to order $n+1$ one has

$$f \circ f^{\circ q} = \lambda z + \sum_{m=2}^{n} b_m z^m + (b_{n+1} + a\,\lambda\,)\, z^{n+1} + \ldots$$

$$f^{\circ q} \circ f = \lambda z + \sum_{m=2}^{n} b_m z^m + (b_{n+1} + a\,\lambda^{n+1}\,)\, z^{n+1} + \ldots$$

Since $f \circ f^{\circ q} = f^{\circ q} \circ f$ the coefficients of these two expressions must agree at all orders, so $\lambda = \lambda^{n+1}$ i.e. $\lambda^n = 1$. Since λ is a primitive q-th root of unity, n must be a multiple of q. □

Finally, a couple of pictures.

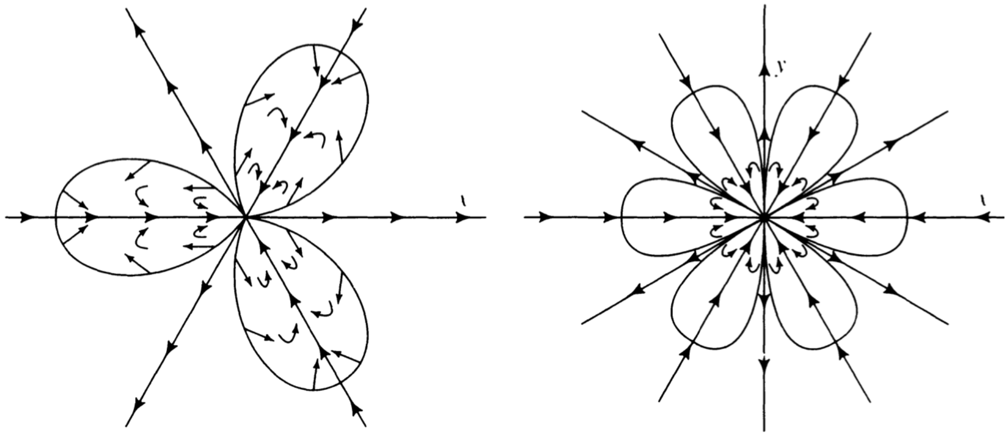

FIGURE 2. Pattern of attracting petals for $z + z^4$ and $-z + z^4$.

Figure 12:

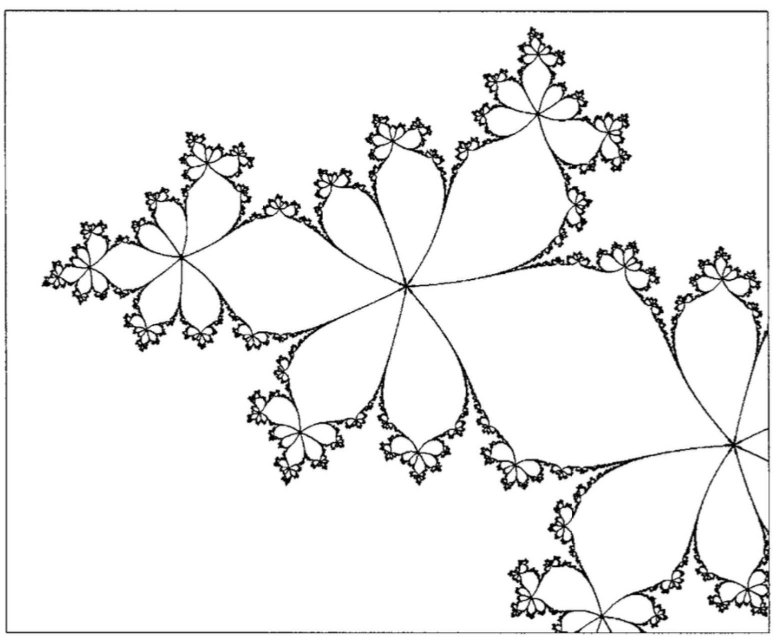

Figure 21. Julia set for $z \mapsto z^2 + e^{2\pi i t} z$ *with* $t = 3/7$.

Figure 13: Top: from [2, p. 40]. Bottom: from [6, p. 109]. Part of the Julia set for a quadratic map f having a fixed point of multiplier $\lambda = e^{2\pi i(3/7)}$ at the origin, near the center of the picture. In this case, the sevenfold iterate $f^{\circ 7}$ is a map of degree 128 with a fixed point of multiplicity $7 + 1 = 8$ at the origin. The seven immediate attracting basins are clearly visible.

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