STRUCTURES
CLUSTER OF
EXCELLENCE

UNIVERSITÄT
HEIDELBERG
ZUKUNFT
SEIT 1386

STRUCTURES JOUR FIXE

Prof. Dr. Gabriele Benedetti
&
Davide Legacci

Mathematisches Institut (Uni HD)

"Zero-sum evolutionary games and convex Hamiltonian systems"

November 20, 2020 1:30 PM

STRUCTURES
CLUSTER OF
EXCELLENCE

UNIVERSITÄT
HEIDELBERG
ZUKUNFT
SEIT 1386

ABSTRACT

Evolutionary games describe the time evolution of large sets of strategically interacting agents who adapt their choices in light of current payoff opportunities. Nowadays they are used to model plenty of phenomena from the emergence of conventions, norms, and institutions in economic, social, and technological environments to natural selection in biological environments.
In this talk we focus on the class of zero-sum replicator games, which model competitive interaction. If there is an equilibrium where all strategies coexist, these games admit a Hamiltonian formulation where the Hamiltonian function is given by the entropy relative to the equilibrium (Akin and Losert, 1984). This is an extremely significant result, which will allow us to use the principle of least action and the convexity of the Hamiltonian to find periodic time evolutions with prescribed relative entropy via numerical optimization.

By ZOOM video webinar system
Meeting ID: 935 6549 3662, Code: 928036
Contact: office@structures.uni-heidelberg.de

Zero-sum evolutionary games and convex Hamiltonian systems

THE PRINCIPLE OF LEAST ACTION

Gabriele Benedetti & Davide Legacci

Presentation of the EP 3.2

STRUCTURES Jour Fixe – November 20, 2020

UNIVERSITÄT
HEIDELBERG
ZUKUNFT
SEIT 1386

Table of contents

Mapping $\mathring{\Delta}^n$ to $\mathbb{R}^n$

Recap

Zero-sum replicator equation with fixpoint point $q \in \mathring{\Delta}^n$ is Hamiltonian with Poisson bracket

$$\{x_i, x_j\} = x_i x_j \Big(\sum_{h=0}^{n} (A_{hj} + A_{ih}) x_h - A_{ij} \Big)$$

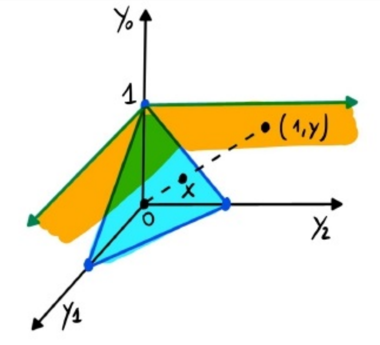
and Hamiltonian function

$$H(x) = \sum_{i=0}^{n} q_i \log \frac{q_i}{x_i} \qquad \text{(entropy of } q \text{ relative to } x\text{).}$$

Step 1: Change x-coordinates to y-coordinates

$$\mathring{\Delta}^n \to (0,\infty)^n, \qquad x \mapsto y,$$

where $(1,y)$ is the intersection between the line through x and the hyperplane $y_0 = 1$.



Step 2: Change y-coordinates to z-coordinates

$$(0,\infty)^n \to \mathbb{R}^n, \qquad y \mapsto z = \log y.$$

New bracket and new Hamiltonian

In the z-coordinates, we have constant Poisson bracket:

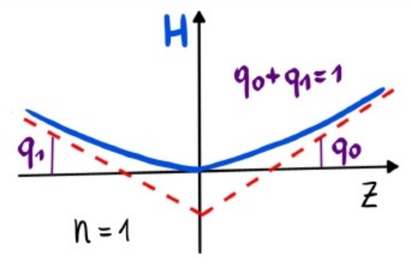
$$\{z_i, z_j\} = B_{ij}, \qquad B_{ij} := A_{0j} + A_{i0} - A_{ij}$$

and Hamiltonian function

$$H(z) = \overbrace{\log\left(1 + \sum_{i=1}^{n} e^{z_i}\right)}^{\text{log-partition function of } x} - \sum_{i=1}^{n} q_i z_i + c.$$

Key facts:

- H convex
- H coercive $\left(H(z) \xrightarrow[|z|\to\infty]{} \infty\right)$



Standard phase space

Step 3: Last (I promise) linear change of coordinates $\mathbb{R}^n \to \mathbb{R}^n$,

$$z \mapsto (q, p, r) \in \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^{n-2k}.$$

The Poisson bracket becomes standard

$$\{p_i, q_j\} = \delta_{ij}, \quad \{p_i, p_j\} = 0 = \{q_i, q_j\}, \quad \{r, \cdot\} = 0.$$

Each $r \in \mathbb{R}^{n-2k}$ is a constant of motion and

$$H_r : \mathbb{R}^{2k} \to \mathbb{R}, \qquad H_r(q, p) := H(q, p, r)$$

is convex and coercive in standard phase space $\mathbb{R}^{2k}$.

Least action and periodic orbits

Clarke's action functional

H_r coercive $\Rightarrow$ the orbits are **recurrent**.
Least action principle detects **periodic** orbits

$$w : \mathbb{R} \to \mathbb{R}^{2k}, \qquad w(t+T) = w(t) \quad \forall\, t \in \mathbb{R}$$

with certain period T and energy $h := H_r(w)$.

Two cases: Find w with **prescribed period** or **prescribed energy**.

For the first case, minimize Clarke's action functional

$$\mathcal{A}(w) := \int_0^T \Big[\tfrac{1}{2}(\dot p \cdot q - \dot q \cdot p) + L(\dot q, \dot p)\Big]\mathrm{d}t, \qquad w = (q, p),$$

where $L : \mathbb{R}^{2k} \to \mathbb{R}$ is the Legendre dual of H

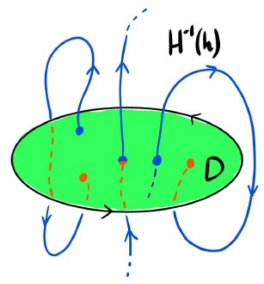
$$L(v) := \sup_{v \in \mathbb{R}^{2k}} \big[v \cdot w - H(w)\big].$$

Three important abstract results

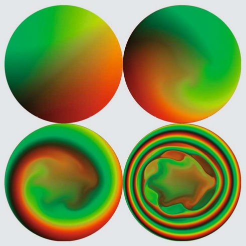
Clarke: The action $\mathcal{A}$ has a minimizer for T larger than some T_0. This minimizer is a periodic orbit with period T.

Weinstein: For all $h > 0$, there is a periodic orbit with energy h.

Hofer–Wysocki–Zehnder $k = 2$: For all $h > 0$, there is a global Poincaré section $D \subset H^{-1}(h)$ for orbits with energy h.



Poincaré section



Iterates of the return map

[Frauenfelder–van Koert]

Research goals

What we want to do

General idea: Implement numerical methods to find periodic orbits of the zero-sum replicator equation.

Concrete goals:

1. Find numerically minimizers of the action functional $\mathcal{A}$.
   Difficulty: non-convex optimization.
2. For $k = 2$ determine numerically a Poincaré section at given energy h and visualize the dynamics.
   Based on: [Frauenfelder–van Koert, 2018].
3. Generalize to a continuous set of strategies:
   E.g. replace $i \in \{0, \dots, n\}$ with $u \in (0, 1)$ and consider

$$\dot{x}(u) = x(u) \int_0^1 A(u - u')x(u)\mathrm{d}u', \qquad A : \mathbb{R} \to \mathbb{R} \text{ odd.}$$

   Difficulty: existence of periodic orbits is unclear.